# PHYSICS COURSE – YEAR 12

**MODULE 5: ADVANCED MECHANICS**

Motion in one dimension at constant velocity or constant acceleration can be explained and analysed relatively simply. However, motion is frequently more complicated because objects move in two or three dimensions, causing the net force to vary in size or direction.

In this module, students will develop an understanding that all forms of complex motion can be understood by analysing the forces acting on a system, including the energy transformations taking place within and around the system. By applying new mathematical techniques, students will model and predict the motion of objects within systems. They will examine two-dimensional motion, including projectile motionand uniform circular motion, along with the orbital motion of planets and satellites, which are modelled as an approximation to uniform circular motion.

As always, teachers are required to provide students with opportunities to engage with all the Working Scientifically skills throughout the course.

I will include some extension sections in these notes. These extensions are not necessary for the completion of this Module.

**PROJECTILE MOTION**

**Inquiry Question:** How can models that are used to explain projectile motion be used to analyse and make predictions?

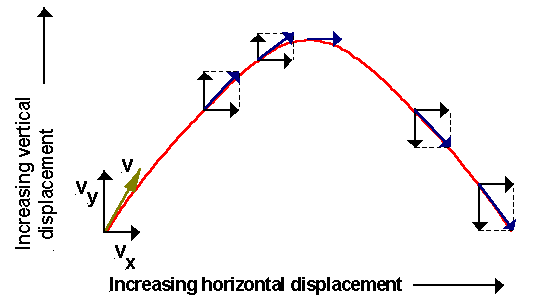
A **projectile** can be any object that has been launched or thrown in a particular direction from some point.  As always in mathematical modelling we start with the simplest system possible that describes the motion reasonably accurately. This means we shall ignore the effects of **air resistance** and assume that the motion is close enough to the surface of the Earth that the **acceleration due to gravity is constant**. This greatly simplifies the mathematics required.

If a projectile moves in a gravitational field and gravity is the only force that acts on it, its path or trajectory is that of a parabola.  Since such a path is two-dimensional, it is convenient to resolve the motion into vector components in the horizontal and vertical directions.  The characteristics of these components are as follows:

* **Horizontal Motion** – a constant velocity motion, the velocity always being the same as the horizontal component of the initial velocity.
* **Vertical Motion** – a uniformly accelerated motion in which the projectile experiences a constant downward acceleration of magnitude g (9.8 ms-2 close to the surface of the Earth).

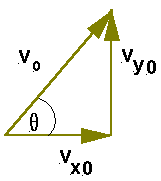
The advantage of resolving the motion into these vector components is that the horizontal and vertical components are completely independent of each other.  Thus, we may apply the equations of uniformly accelerated motion from Module 1 to each component separately and then add the two components together as vectors to obtain the actual motion of the projectile.

The following diagram illustrates this idea.



Clearly, the horizontal velocity remains constant throughout the motion, as indicated by the horizontal velocity vector staying the same length.  The vertical velocity decreases as the projectile moves upwards, is zero at the maximum height attained by the projectile and then increases again as the projectile returns to the ground.  This is due to the constant downwards acceleration due to gravity.  Note that at any time we can calculate the total velocity by adding the vertical and horizontal vector components together.  The total velocity vectors are tangents to the trajectory.

If we assume the projectile whose trajectory is shown in the diagram above, was launched at velocity **v0** at an angle ****to the horizontal, we can construct the following vector diagram representing these initial conditions.



The zero subscripts attached to the velocity vectors denote that this is the initial velocity of the projectile, that is the velocity at time t = 0.  Using simple trigonometry, we see that:

**vyo = vosin**

**vxo = vocos**

Note that these velocities are the initial vertical and horizontal velocities of the projectile.  **The velocities at any time t after t = 0 are found by applying the appropriate uniformly accelerated motion equation: v = u + at.**

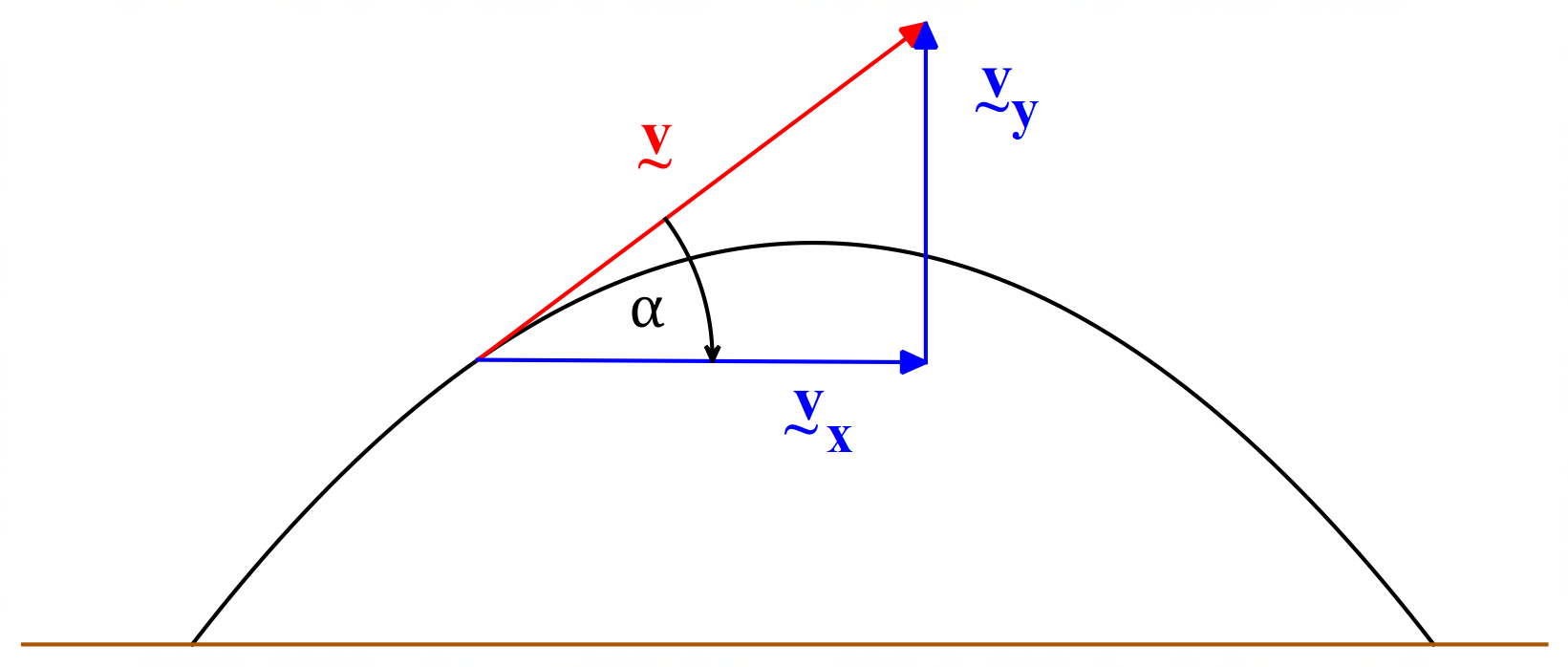
**vy = vosin gt**

**vx = vocos    (since vx remains constant)**

So, the magnitude and direction of the total velocity at any time t is given by:

**V =** ** (vx2 + vy2)  and  = tan-1(vy / vx)**

where **** is the angle made by the total velocity vector and the horizontal, as shown below.



**The displacement of the projectile at any time t is found from: s = ut + ½ at2:**

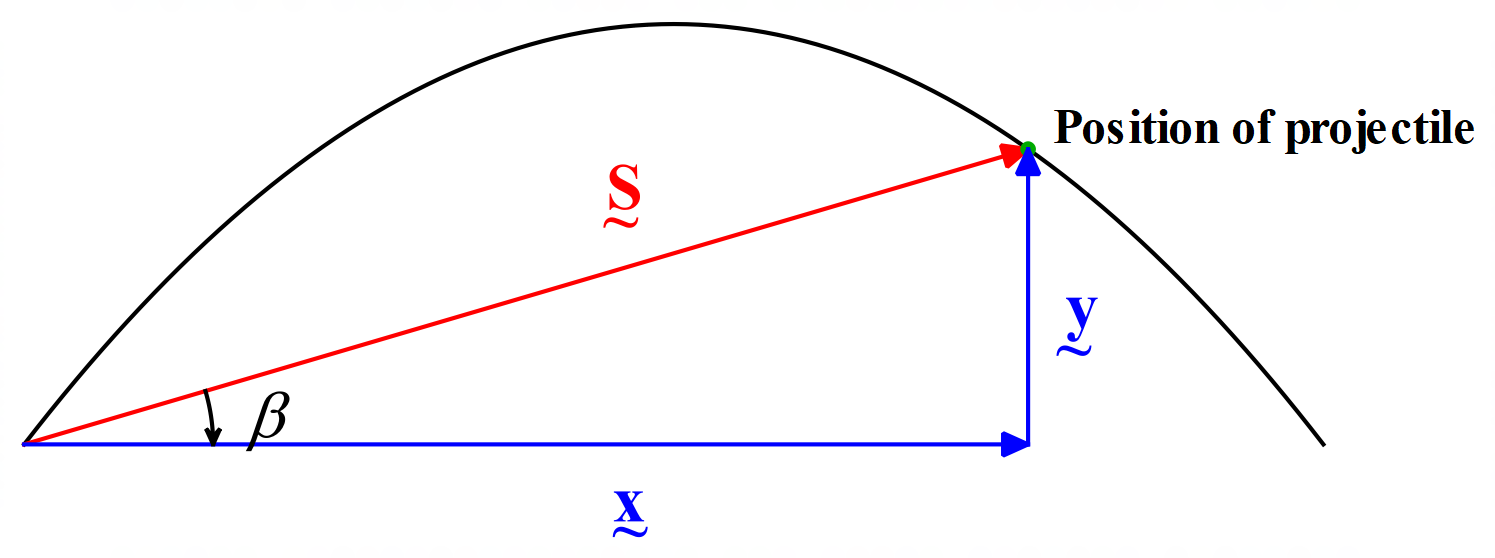
**y = vosint  ½ gt2**

**x = vocos.t**

So, the magnitude and direction of the total displacement at any time t is given by:

**S =** ** (x2 + y2)  and  = tan-1(y/x)**

where **** is the angle made by the total displacement vector and the horizontal, as shown below.



The fact that the trajectory of the projectile is a parabola can be shown by using the displacement equations (for y and x) given above and eliminating t from the equations.  This gives the standard equation of a parabola.  Try this as an exercise.

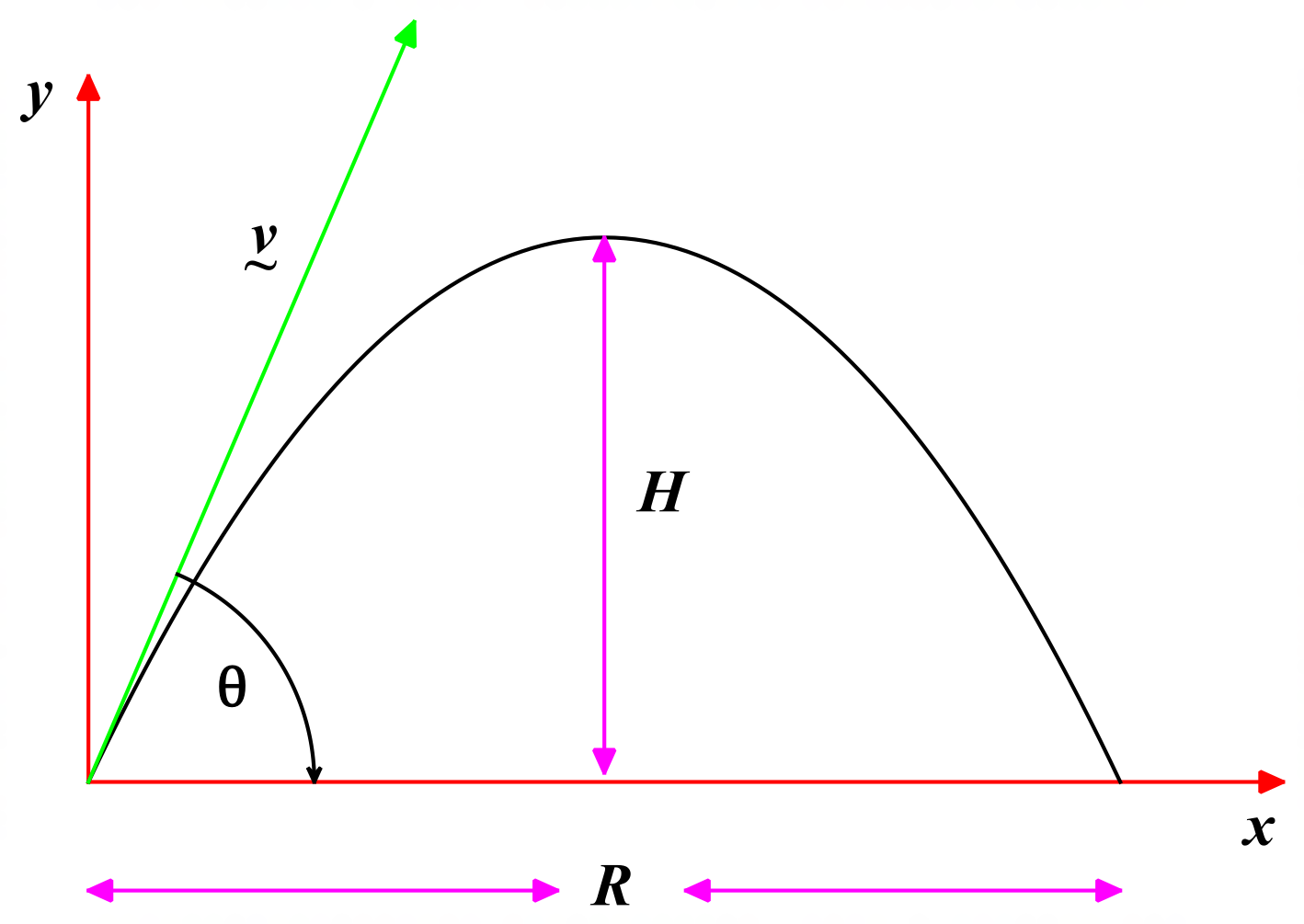
**Solving Projectile Motion Problems**

The equations derived above are sufficient to solve all numerical projectile motion problems that you will come across in this course. You should not need to learn them off by heart because they are just the uniformly accelerated motion equations you learnt to use in Module 1, applied to the case of a projectile. You can recreate the above equations whenever you need them.

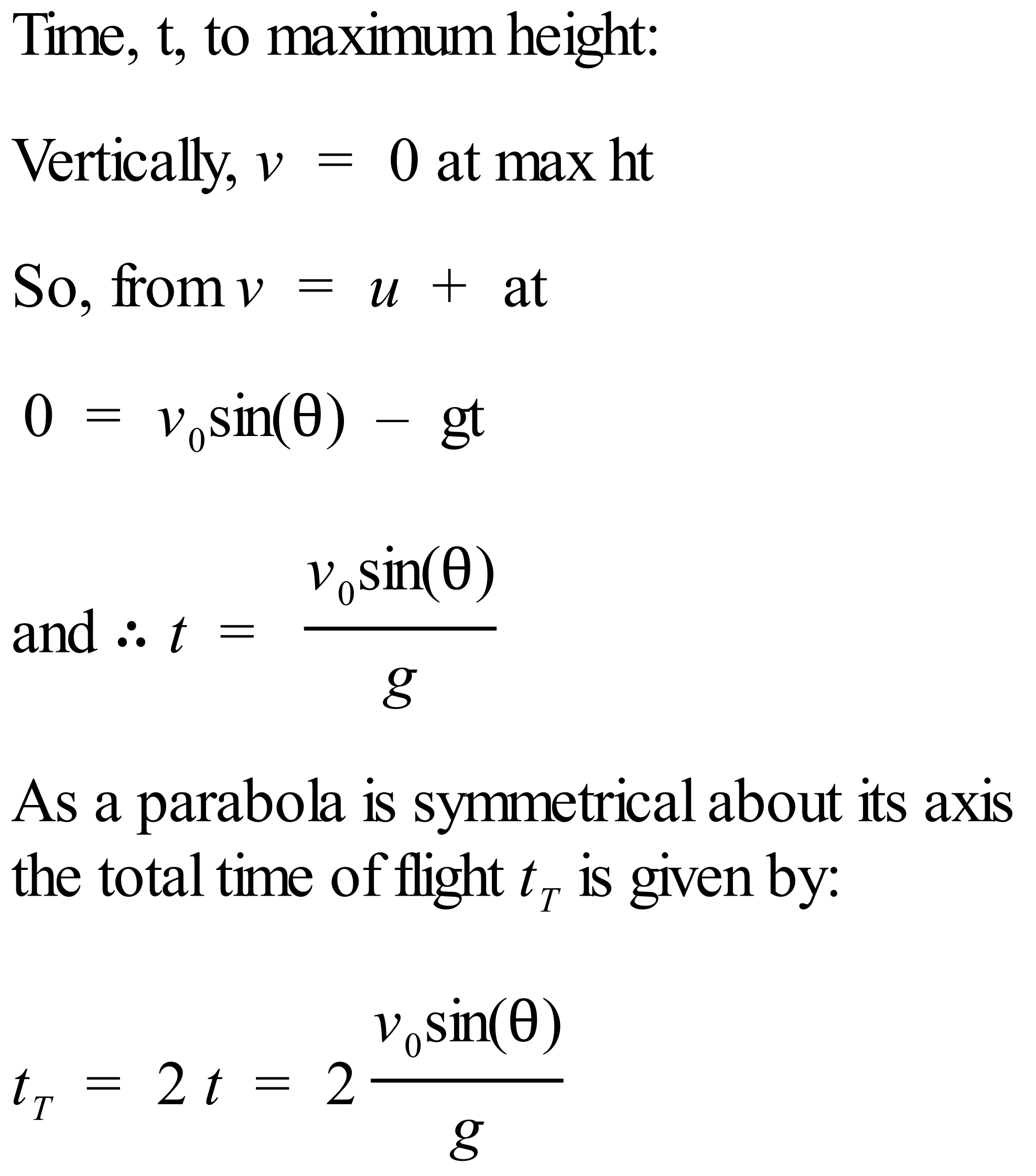
For each situation, think carefully about what you know. For instance, if you are asked to find the time taken for a projectile to reach its maximum height, you know that at the maximum height, the **vertical** velocity is zero. So, you could use **v = u + at** vertically. You know that **v** = 0; you would normally be given the initial velocity, **vo ,** and launch angle, ****, for the projectile, from which you can determine **u** in the vertical direction (**vosin**). And **a** = **g** = - 9.8 ms-2 (note the minus sign, indicating that the acceleration is acting in the opposite direction to the initial vertical velocity). Thus, you can calculate the time to the maximum height, **t**. If you were then asked for the total time of flight, you would remember that a parabolic path is symmetrical about its axis. So, if the time to the maximum height is **t**, then the total time of flight will be twice this, **2t**.

**Equations for Time of Flight, Range & Maximum Height**

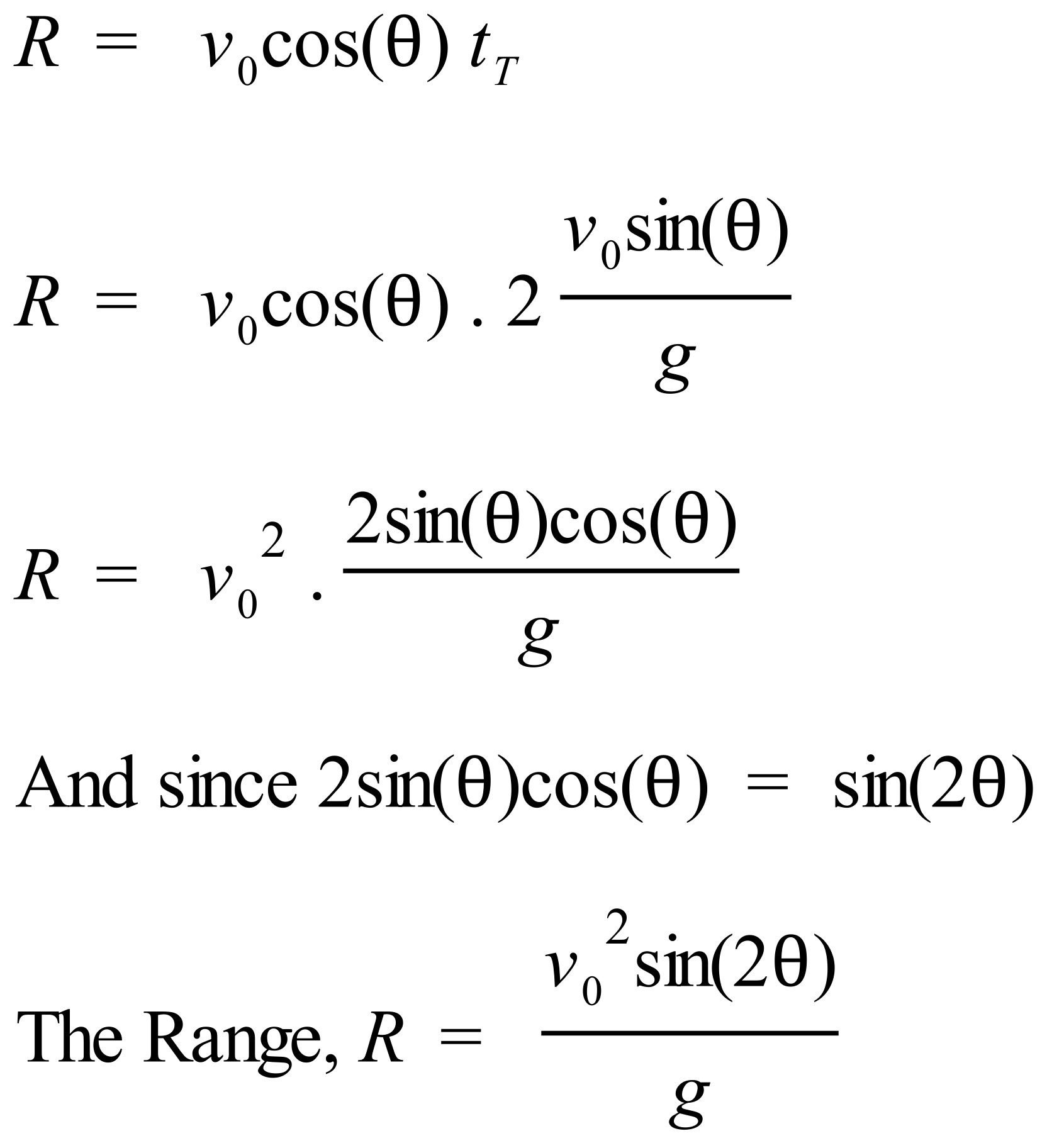
Our mathematical model of projectile motion can be used to derive further useful equations, for time of flight, maximum height reached and range of the projectile. The term range refers to the maximum horizontal distance travelled by the projectile. The derivations follow below.



In the diagram above, a projectile is launched with velocity **v̰** at an angle **** to the horizontal. The maximum height reached by the projectile is **H** and the range of the projectile is **R**.



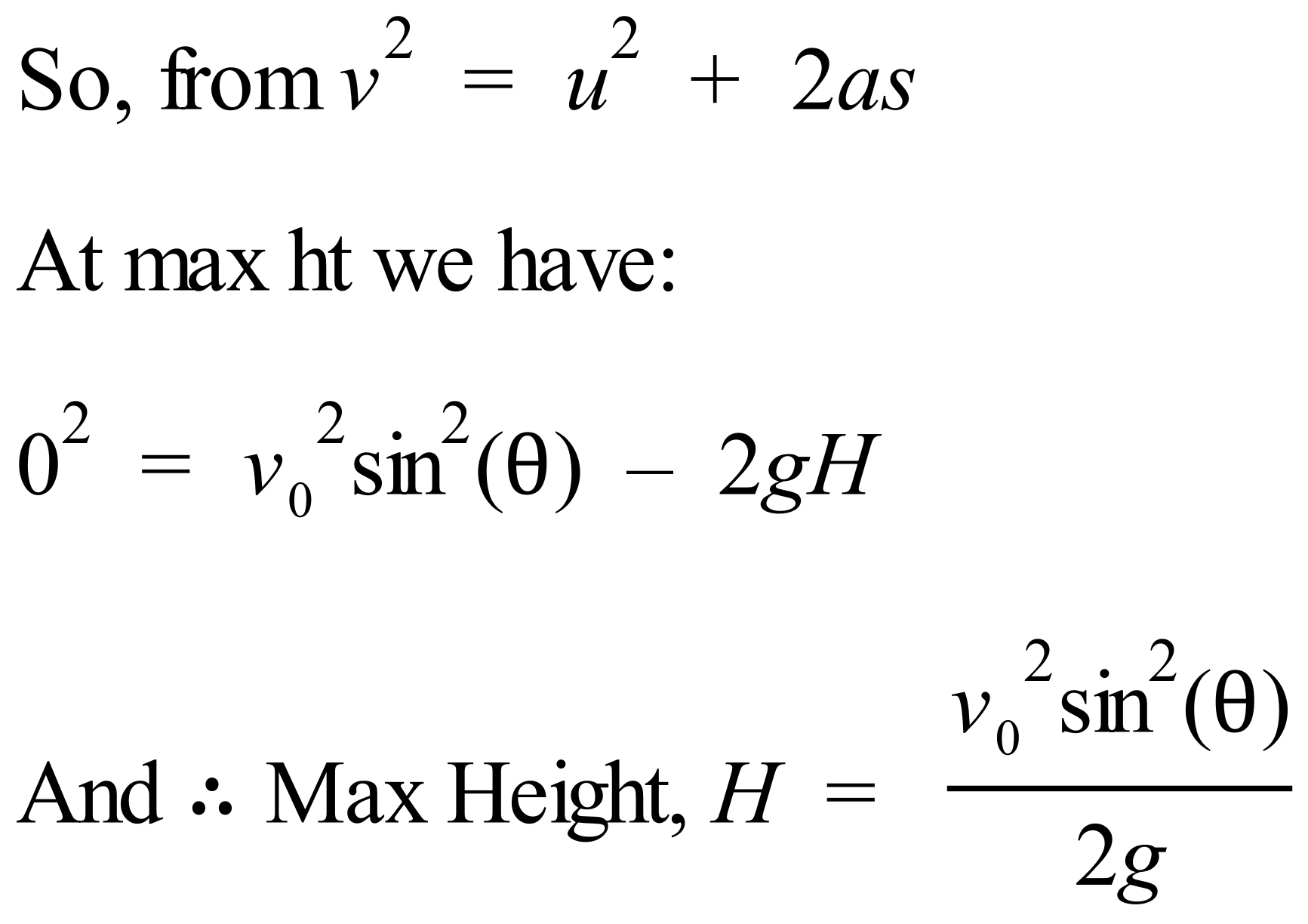
Now let us consider, the range of the projectile. As the horizontal velocity of the projectile remains constant throughout the motion (due to us ignoring air resistance), the range of the projectile is given by **x = vocos.t**, as we saw earlier, where **t** is the total time of flight. This can then be re-written using the result for the total time of flight shown above.



Note that the maximum value of sin θ is 1, when θ = 90°. So, the maximum value of sin 2θ occurs when θ = 45°. So, the **maximum range** of a projectile occurs if it is launched at 45° to the horizontal.

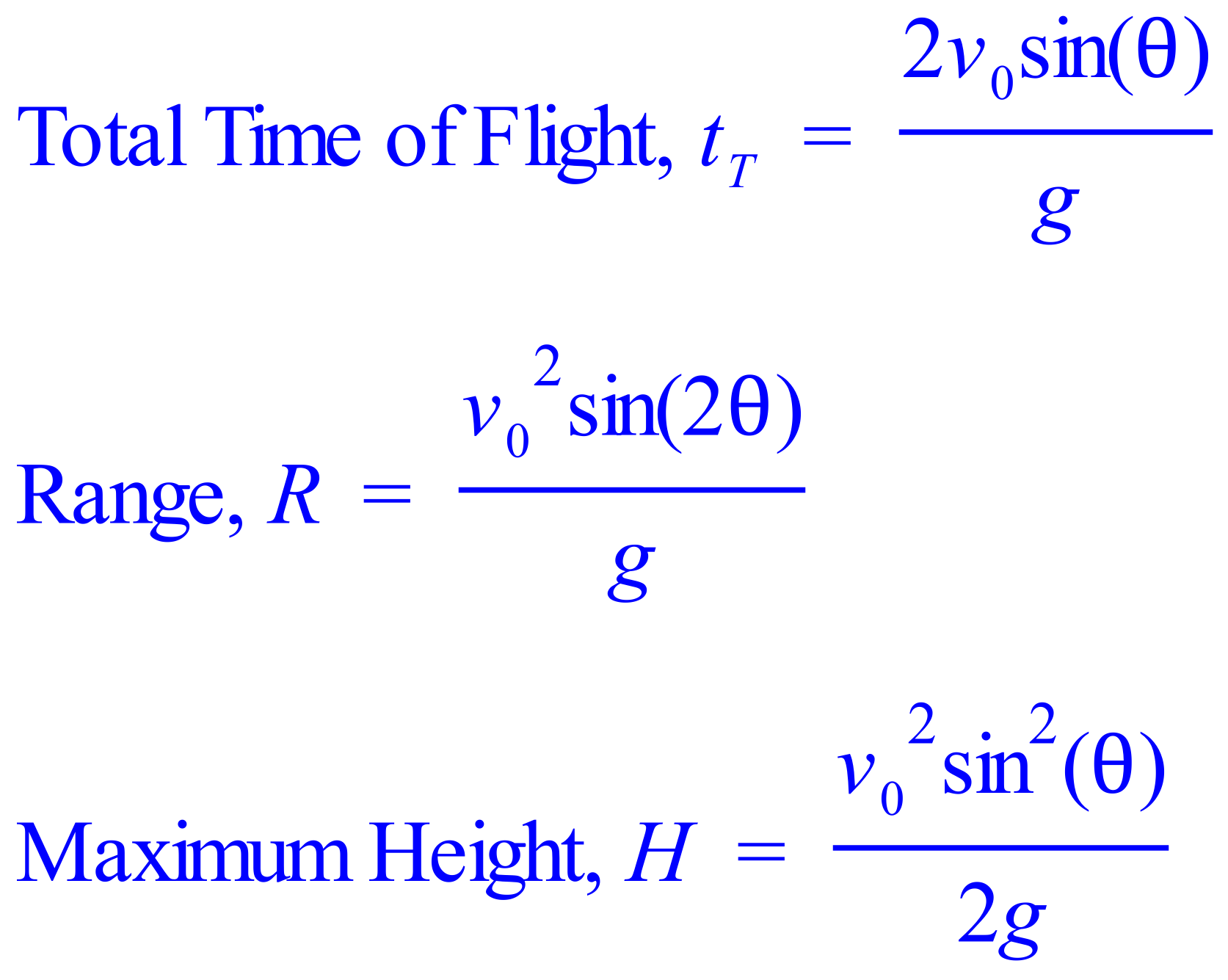
Check out these projectile motion apps on [Phet](https://phet.colorado.edu/en/simulation/projectile-motion).

To derive the equation for maximum height, H, we use the fact that at max height, the final vertical velocity is zero.



Note carefully that in this last equation the sine function is squared. You must take the sine of the angle and then square that answer. That is what sin2() means.

So, in summary our equations for **time of flight, range and maximum height reached** by a projectile are as follows.



Mathematical models are powerful tools for deriving relationships between physical variables. Remember, the equations we have derived apply when we ignore air resistance and assume that the motion occurs close to the Earth’s surface, so that the acceleration due to gravity is constant.

Try some of the problems provided on the Module 5 web page.

Your teacher should provide opportunities for some practical work on projectile motion.

**Extension (Not necessary for completion of this Module): Galileo’s Analysis of Projectile Motion**

Our understanding of projectile motion owes a great debt to Galileo, who in his work entitled “Dialogues Concerning Two New Sciences”, presented his classic analysis of such motion.  Galileo argued that projectile motion was a compound motion made up of a horizontal and a vertical motion.  The horizontal motion had a steady speed in a fixed direction, while the vertical motion was one of downwards acceleration.  Using a geometric argument, Galileo went on to show that the path of a particle undergoing such motion was a parabola.

In his work Galileo admits that his assumptions and results are only approximations to the real world.  He admits that due to resistance of the medium, for instance, a projectile’s horizontal motion cannot be truly constant in speed.  He states quite clearly that in reality the path of the projectile will not be exactly parabolic.  He argues, however, that his approximations can be shown by experiment to be close enough to the real world to be of very real use in the analysis of such motion.  In doing this, he became perhaps the first scientist to demonstrate this modern scientific attitude.  His approach was certainly very different from that of the ancient Greek geometers, who were only interested in exact results.  A translation of Galileo’s analysis of projectile motion can be read at:

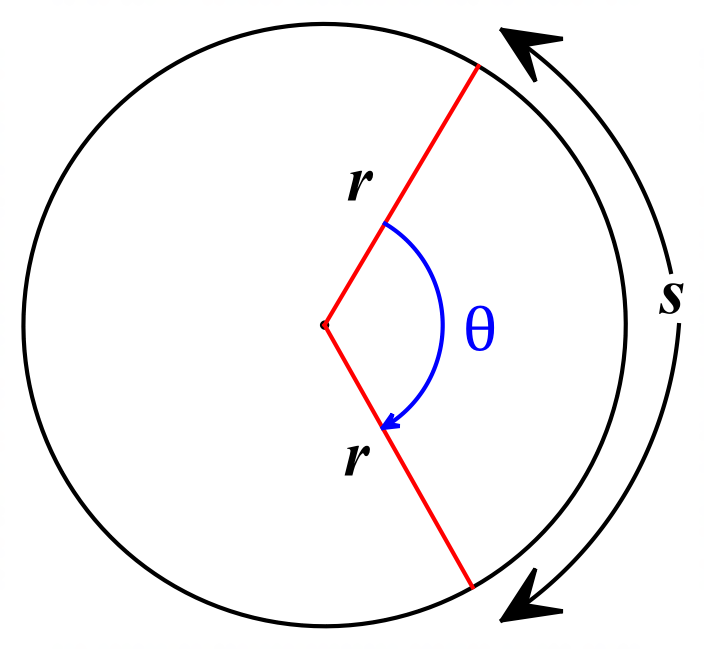
[**http://galileo.phys.virginia.edu/classes/109N/tns244.htm**](http://galileo.phys.virginia.edu/classes/109N/tns244.htm)

**CIRCULAR MOTION**

**Inquiry Question:** Why do objects move in circles?

Well, I am a little amused that this was the best inquiry question that the syllabus writers could come up with for circular motion. They would have been better to use the same question as for projectile motion & just change the word “projectile” to “circular”. Anyway, let’s move on.

In order to study motion in a circle we require an alternative measure of angle to the “degree”. The reason is to simplify the units (and maths) associated with circular motion. As many of you would know already, from your study of mathematics, the **radian** is defined as the ratio of the length of the arc of a circle, s, to the length of the radius of the circle, r, where **each length is measured in the same unit**.



By definition, **θ = s/r**

Since **s & r** are both measured in metres, the radian measure, **θ** , is a dimensionless quantity. It has no units. Symbols used to indicate radian include “rad” or a lowercase “c” as an index written as below.

**1 radian = 1 rad = 1c**

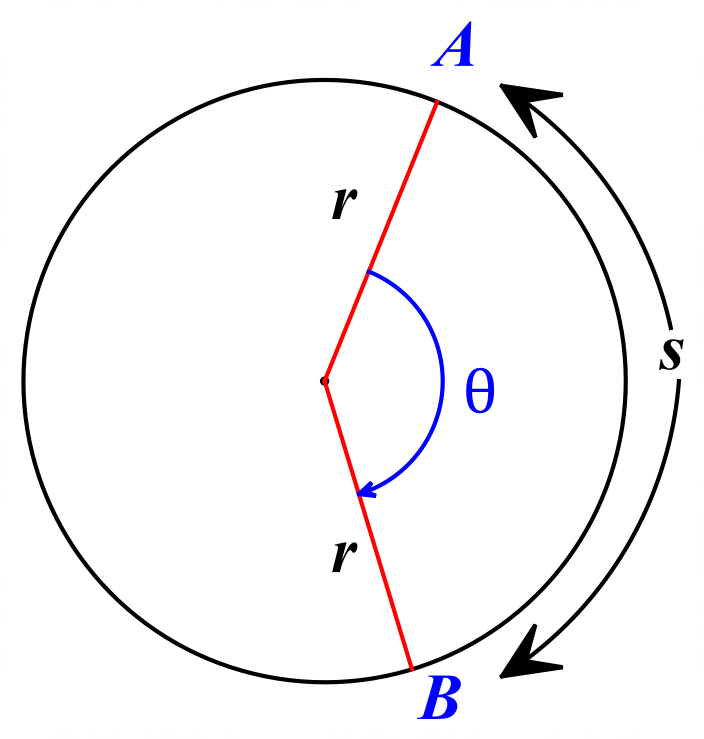
If no symbol is used, we always assume radians. If an angle is ever expressed in degrees, it must be written with the degree symbol.

Clearly, 1 radian is the angle subtended at the centre of a circle by an arc whose length is equal to the radius. θ = s/r = 1, since s and r are equal in this case.

If you would like to delve in more detail into the reasons why the use of radians is beneficial to the study of circular motion, try [this website](https://qedinsight.wordpress.com/2011/03/14/why-radian-measure-makes-life-easier-in-mathematics-and-physics/).

**Angular Displacement**

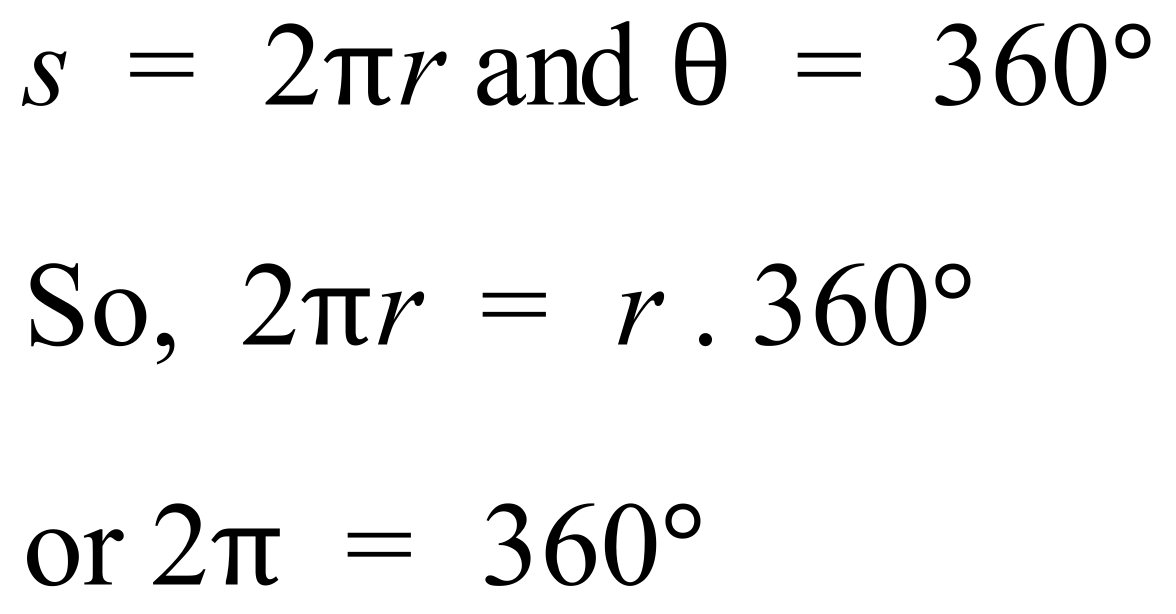
The angular displacement of one point, A, from another point, B, on a circle, is the angle in radians through which you have to move to go from A to B.



So, by definition, the **angular displacement,** **θ = s/r**.

Thus, we have that the **arc length** between two points on a circle is **s = rθ**.

From this we can determine that if the arc length is equal to the circumference of the circle, then:



Thus, we have a method for quickly converting degrees to radians and vice versa. Use proportionality ratios.

eg to convert 45° to radians: 45/360 = x/2solve for x and we have x = /4. So, 45° is the same angle as /4 radians.

Many of you will already be familiar with radian measure. Don’t panic if you are not. You will pick it up very quickly. You will soon know off by heart the radian equivalents to the angles with which you are familiar in degrees: 30°, 45°, 60°, 90°, 180°, 270° and 360°. If you are not familiar with them already, it would be sensible to use the method above to find the radian measure of each of these commonly used angles. (They are, in the same order as above: and

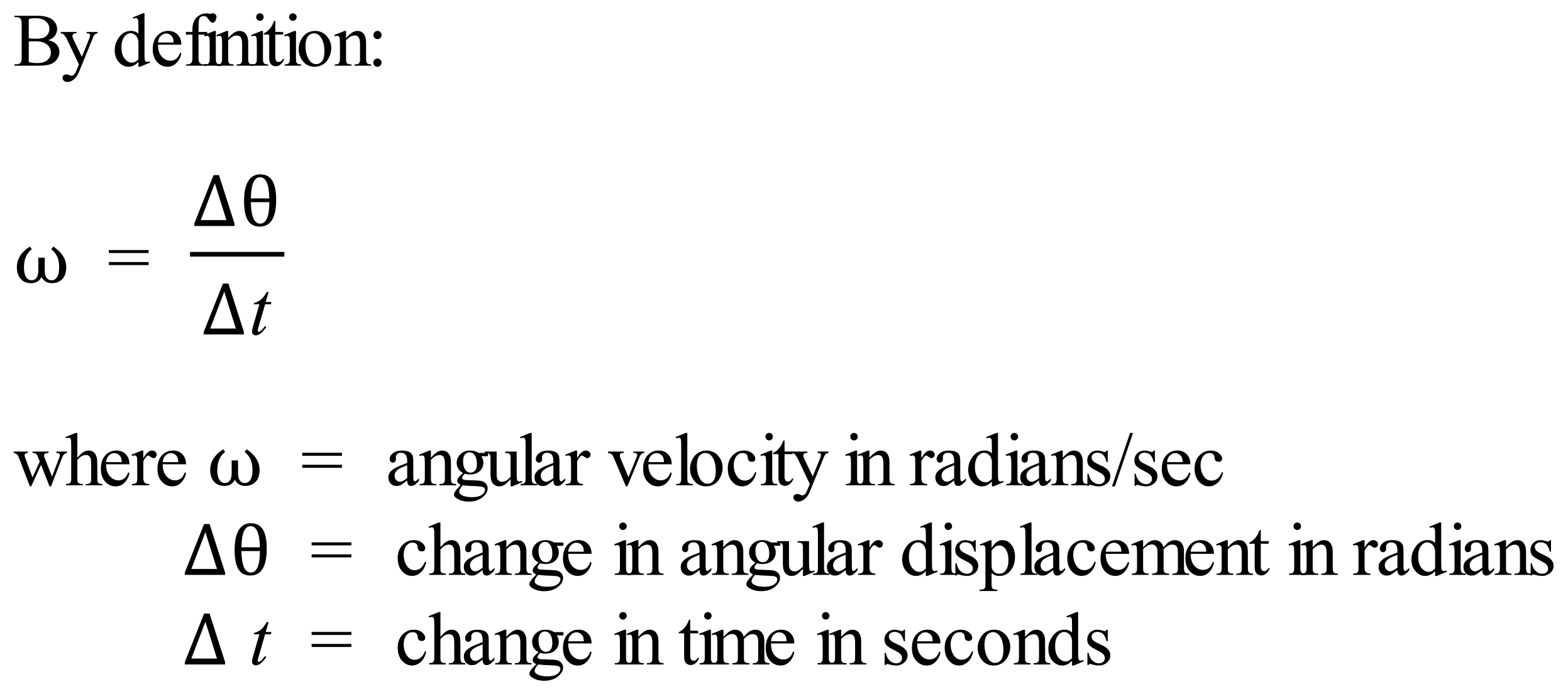
**Special Note on Direction of Angular Quantity Vectors**

**Vectors must have magnitude and direction.** The direction of angular quantity vectors (eg angular displacement) points **perpendicular to the plane of the** **motion**. You can determine this direction using the right-hand rule: Curl the fingers of your right hand in the direction of angular motion. The direction in which the thumb of your right hand points, is the direction of the angular quantity vector. **You will not be required to state the direction of angular quantity vectors in this Stage 6 course.** I have included this information for completeness.

Note that in uniform circular motion, the direction of angular quantity vectors remains constant. However, the direction of **linear quantity vectors** such as **linear velocity** and **linear momentum** changes as an object moves in UCM.

**Angular Velocity and Uniform Circular Motion (UCM)**

When an object moves in a circle, its angular displacement changes with time. This change in θ with t is called the **angular velocity** of the object. It is usually given the symbol ****the lowercase Greek letter omega.

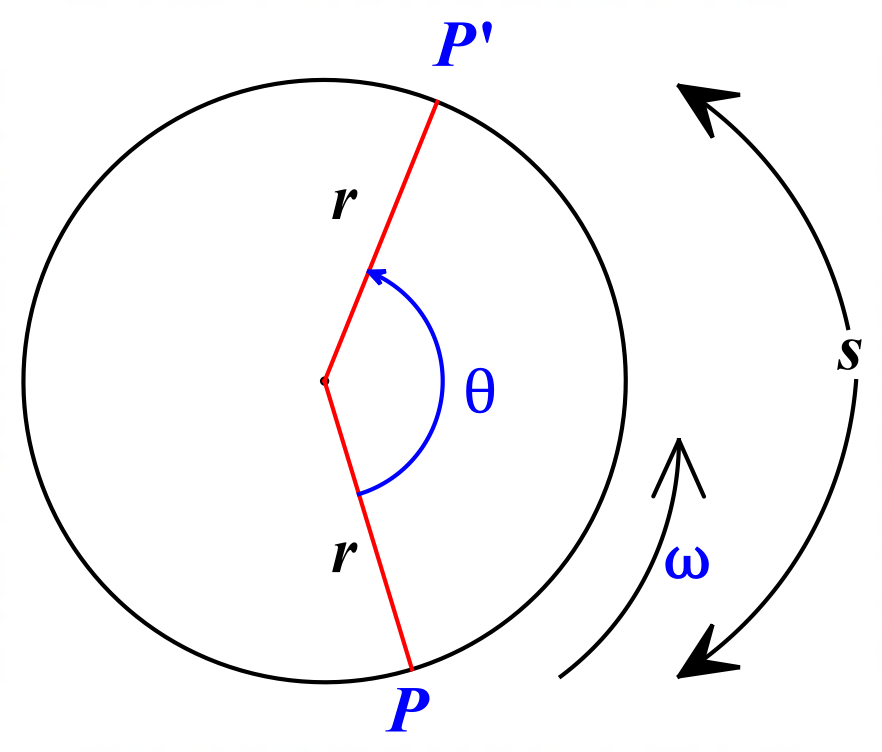


An object sweeping out equal angles in equal periods of time is said to be executing **uniform circular motion (UCM)**. An object executing uniform circular motion therefore has a constant angular velocity, ****

Note: The equation for angular speed given in the Physics Stage 6 Syllabus is . This expression should always be written as .

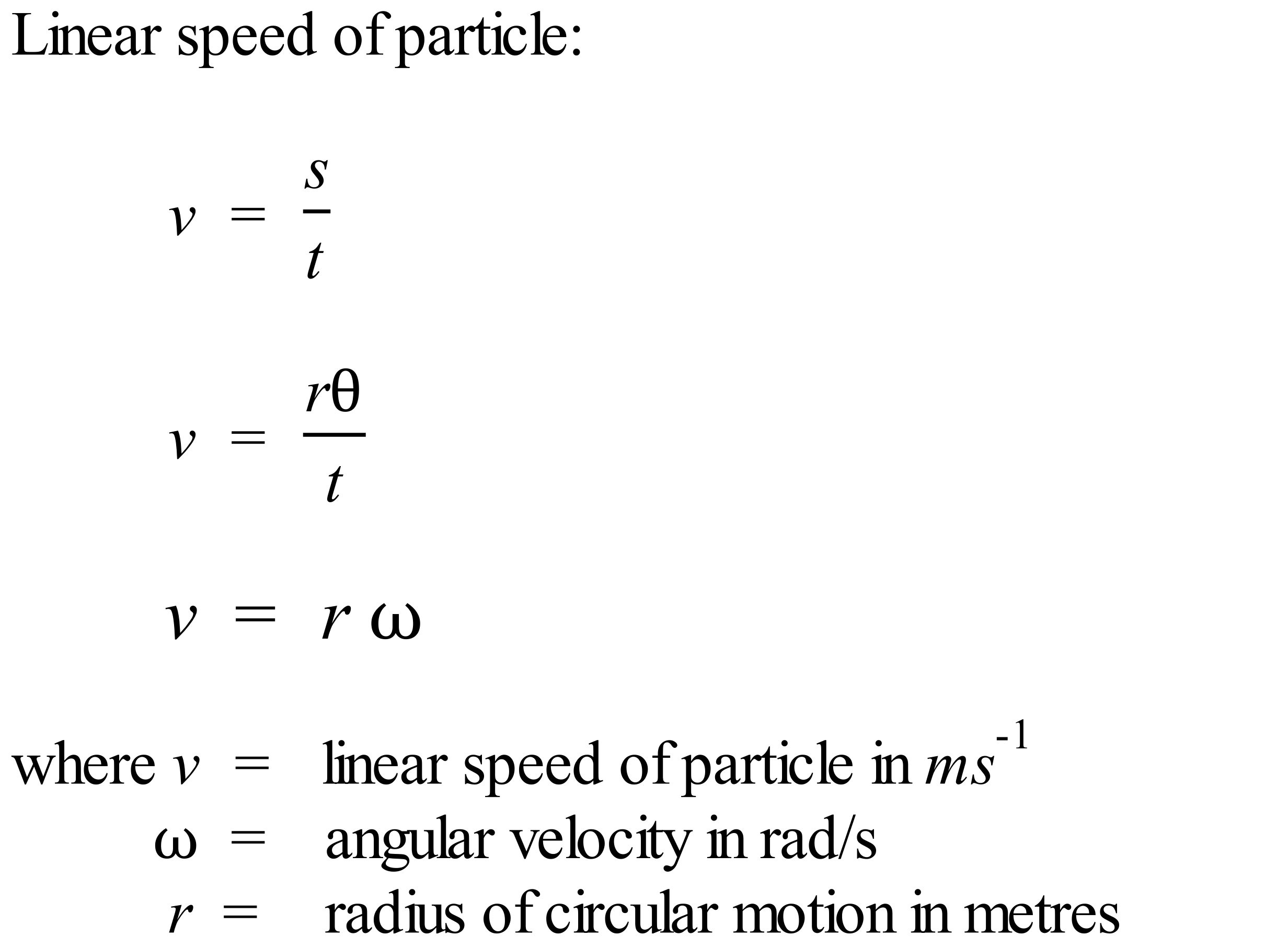
**Linear Velocity of Particle Executing UCM**

Consider a particle P executing uniform circular motion with angular velocity **** as shown below. In a time t, P moves through an angular displacement of θ radians.



Linear speed = distance moved / time taken

Hence, we have:

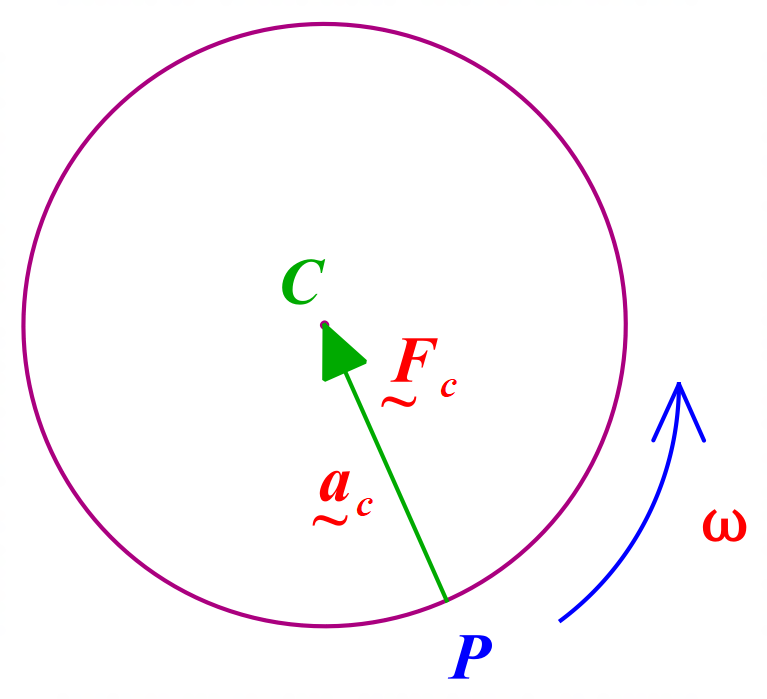


By stating the direction of this linear speed at any time t, we have determined the **linear velocity** of the particle at that time. Clearly, the direction of linear velocity changes as the particle moves around the circular path. **The direction is always a tangent to the circle.**

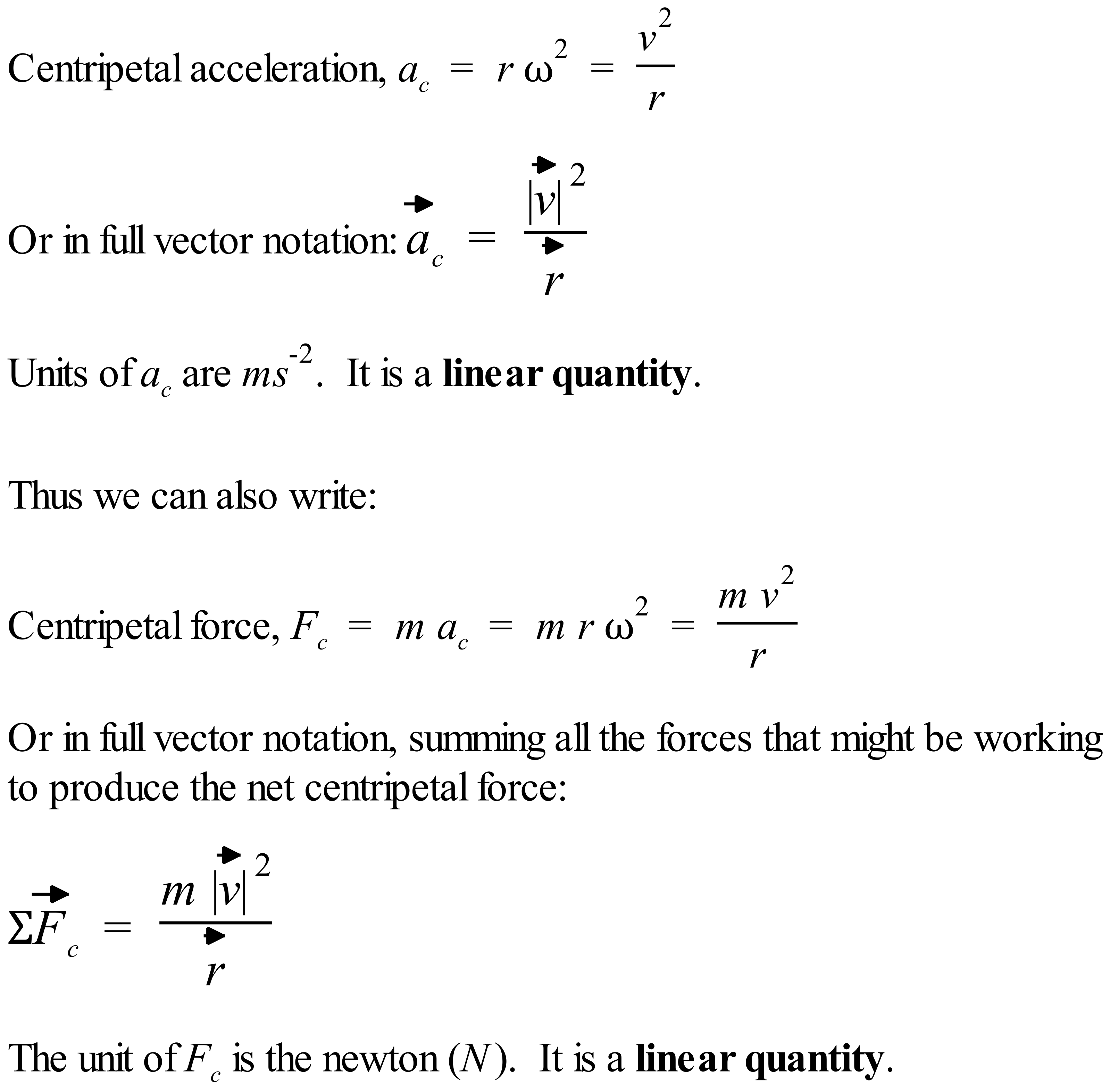
Note that the above derivation is best done using calculus rather than the approach taken above.

**Centripetal Acceleration and Centripetal Force**

The acceleration and force acting on a particle executing uniform circular motion always act towards the centre of the circular motion. For this reason, the term **“centripetal” (centre-seeking)** is applied to both the acceleration and the force. See diagram below. The vector pointing towards the centre of the circle is showing the direction of both the centripetal acceleration and centripetal force vectors.

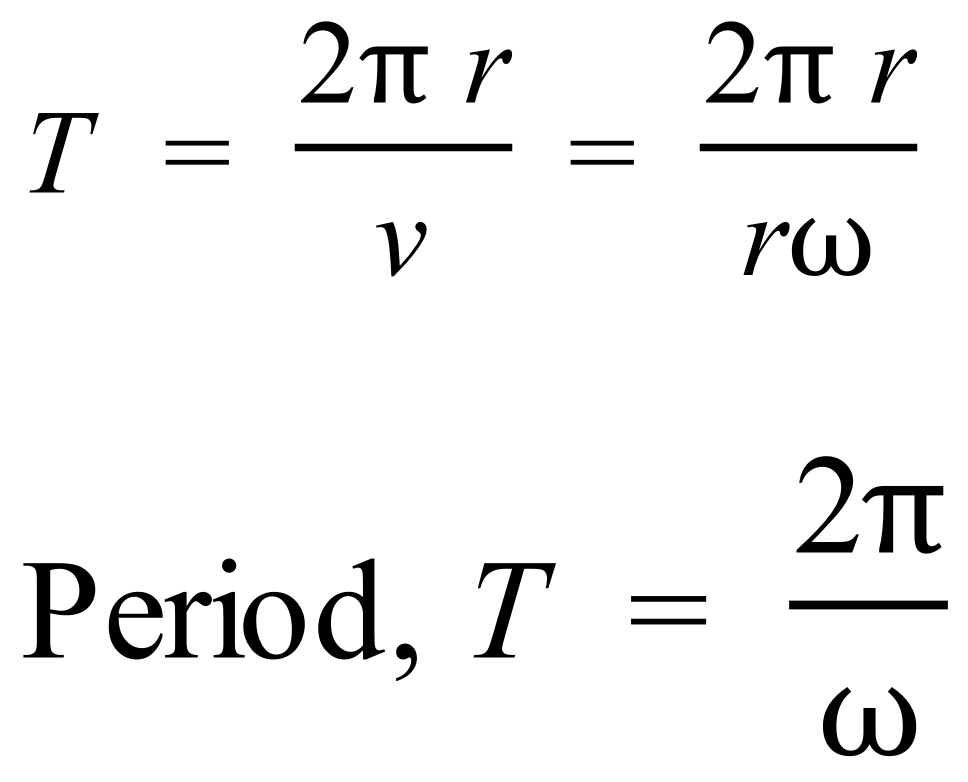


It can be shown (again using calculus is best) that:

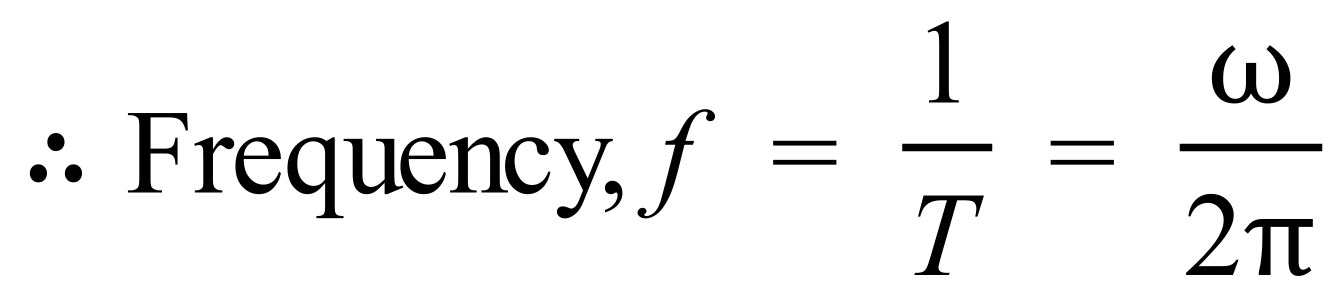


**Period and Frequency**

The **period, T,** of a particle executing UCM is the time taken for the particle to make one complete revolution. The unit of period is the second (s).



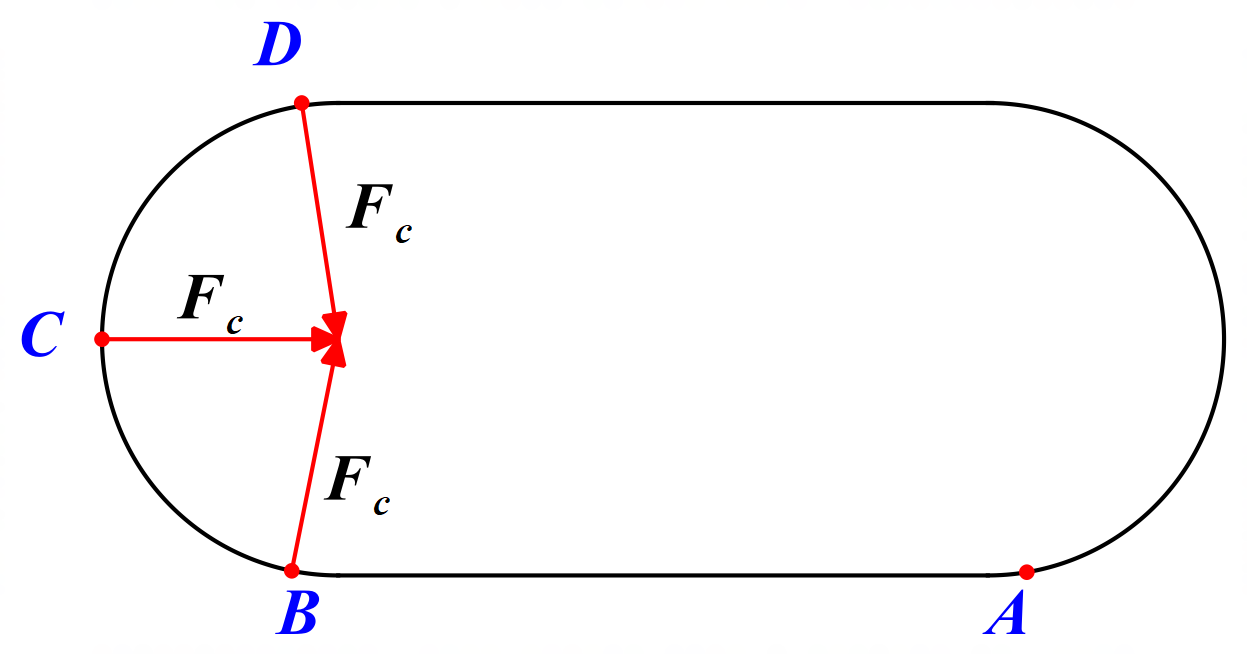
The **frequency, f,** of a particle in UCM, is the number of times per second that it completes a full cycle of its motion. The unit of frequency is the hertz (Hz) or cycles per second (cycles/s) or just per second (s-1).



**Examples of Objects Executing UCM**

Let us analyze the forces acting on objects undergoing UCM in a variety of situations.

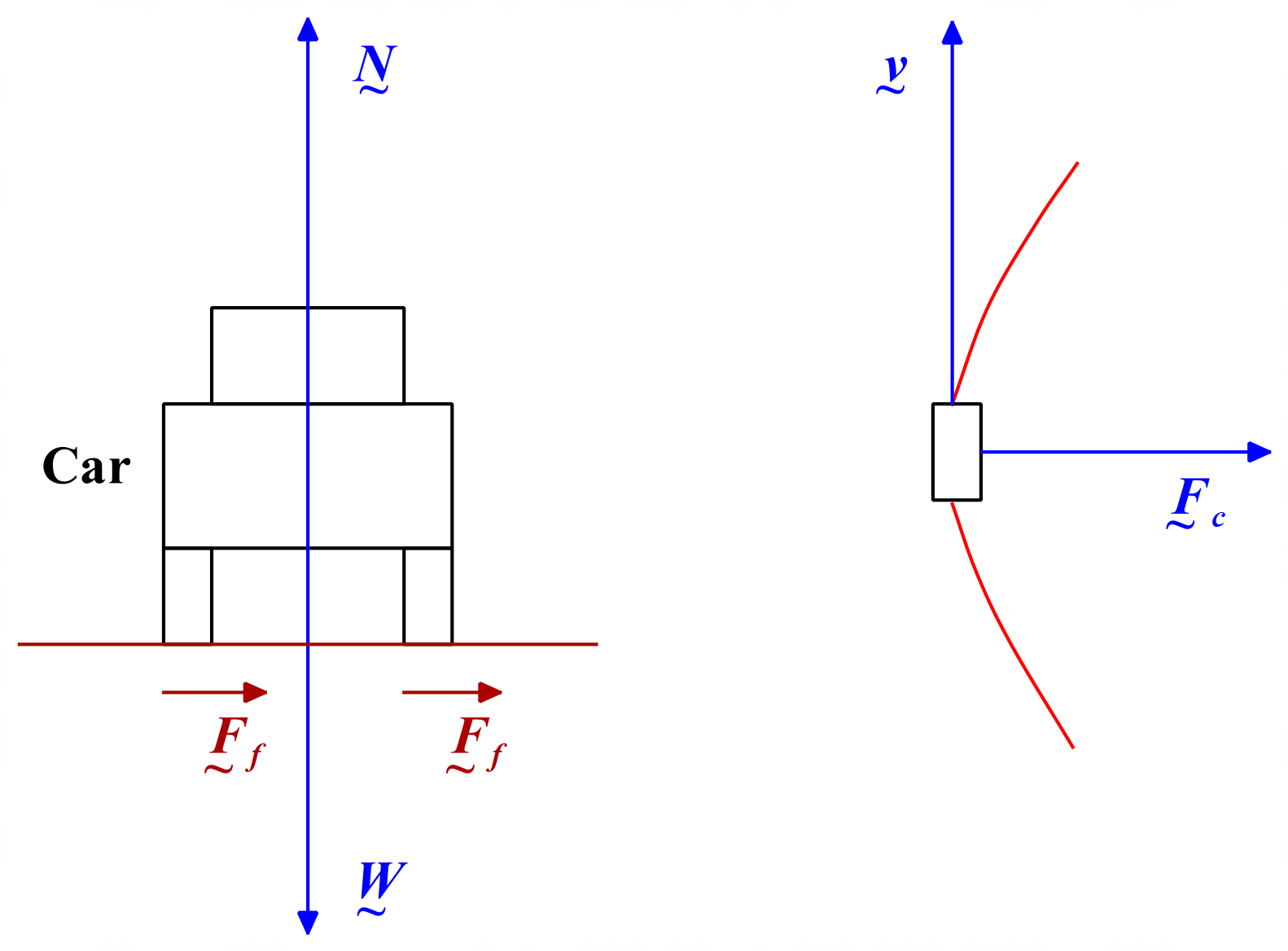
**1. Car Moving Around Horizontal Circular Bend**



Imagine looking down from above on a race track shaped as shown. Two long straight sections lead into two circular curved sections. The track is completely level. The car starts from A and travels at a constant speed towards B.

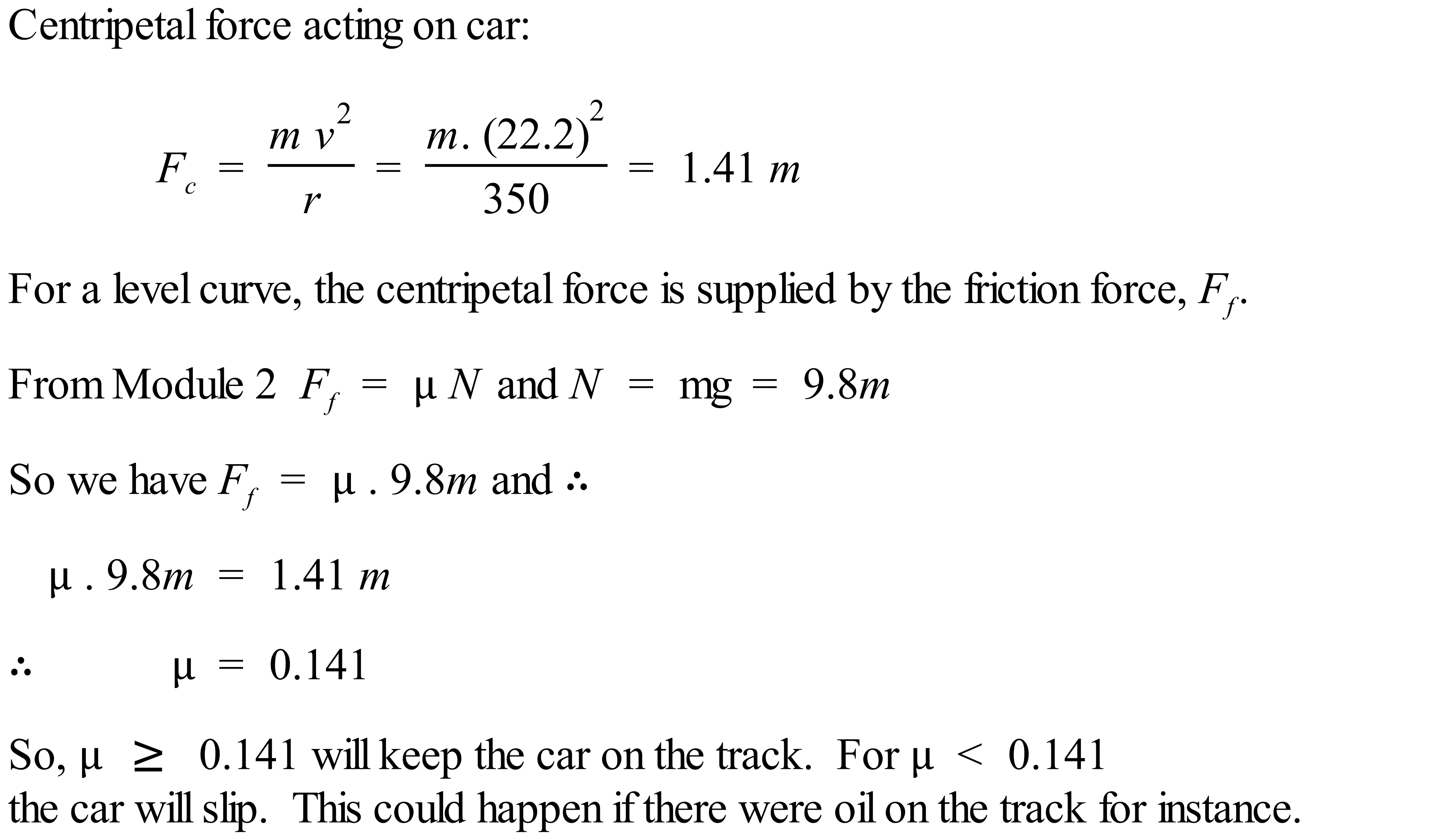
As the car reaches B, it begins to go around the circular bend. Assuming the car stays on the track, there must be a force acting on the car towards the centre of its circular path to hold it in circular motion. The only place this force can come from is the friction force between the tyres and the road. No other part of the car is capable of supplying this force which is acting horizontally and perpendicular to the forward motion of the car. See diagram below, which depicts the car at point C.





In diagram above we have a free body diagram on the right looking down on the car moving around the bend and showing the net centripetal force, Fc , on the car and the linear velocity, v, of the car. On the left we have a cross-section diagram of the car (yes, I am no artist). The force of friction, Ff, between each tyre and the road is the only horizontal force acting on the car and provides the centripetal force. The normal force, N, and the weight force, W, have no components in the horizontal direction and therefore cannot contribute to the centripetal force.

As a quick numerical example, let’s assume that in the above scenario the car is moving at 80 km/h (22.2 m/s) around a curve of radius 350 m. We could then calculate the coefficient of friction required to enable the car to stay on the track at that speed. Assume mass of car = m.



**Concept of Pseudoforces**

If you were a passenger in the car in the example above, and you were not tightly strapped into your seat, you may be excused for thinking that as you go around the bend there is a force acting on you that throws you towards the outside of the curve, away from the centre of the circular path. This force is often referred to as a centrifugal (centre-fleeing) force. Its direction is opposite to the centripetal force holding the car in UCM.

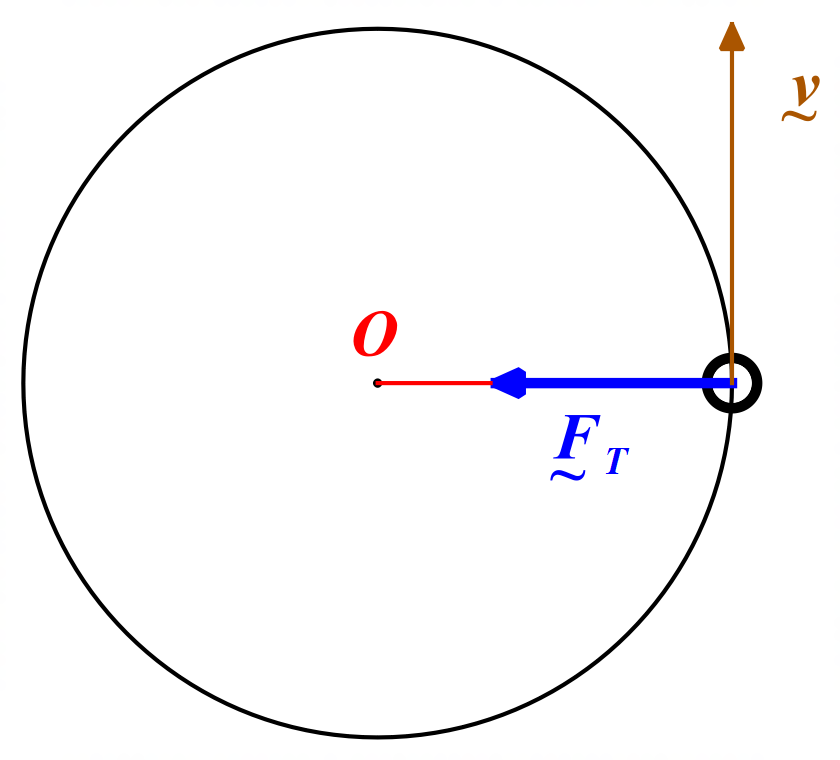
**This centrifugal force is not real.** It is a fictitious force, usually referred to as a **pseudoforce**.

Let’s consider what is happening. As the car begins to traverse the bend, by Newton’s 1st Law you continue to move in the same direction you were originally heading for a short period of time. During this time the car turns into the bend and the door runs into your shoulder as your shoulder travels in its original direction of motion. To you, it seems like a force has acted and thrown you against the door. In reality, someone watching closely from outside the car would observe that you continued to move in a straight line while the car turned under the influence of the centripetal force acting towards the centre of the circular path. They would claim that it was the door under the influence of the centripetal force that ran into your shoulder. The pseudoforce (centrifugal force) is not necessary to accurately explain the physics of the situation.

All manner of examples of pseudoforces exist – every time you go on a ride that rotates (eg a rotor) at an amusement park, for instance. In all cases, the pseudoforces can be shown not to be necessary to accurately describe the situation.

**2. A Mass on a String**

Our second example is a mass on the end of a string swung in a horizontal circle.



The diagram above represents a ball on the end of a rope that is anchored to the centre, **O**, of a flat, level table. The ball is swung in a horizontal circle on the surface of the table. In such a case, the **tension in the rope, FT,** supplies the force on the ball that keeps it moving in a circular path – the centripetal force. The circular motion occurs because the ball is travelling fast enough that each time the centripetal force pulls the ball towards the centre of the circle, its linear velocity, **v**, directed tangentially to the circle, moves it slightly further around the circular path. The forces in the vertical direction – the normal force on the ball upwards and the weight force on the ball downwards – do not contribute to the circular motion as they have no components in the horizontal direction.

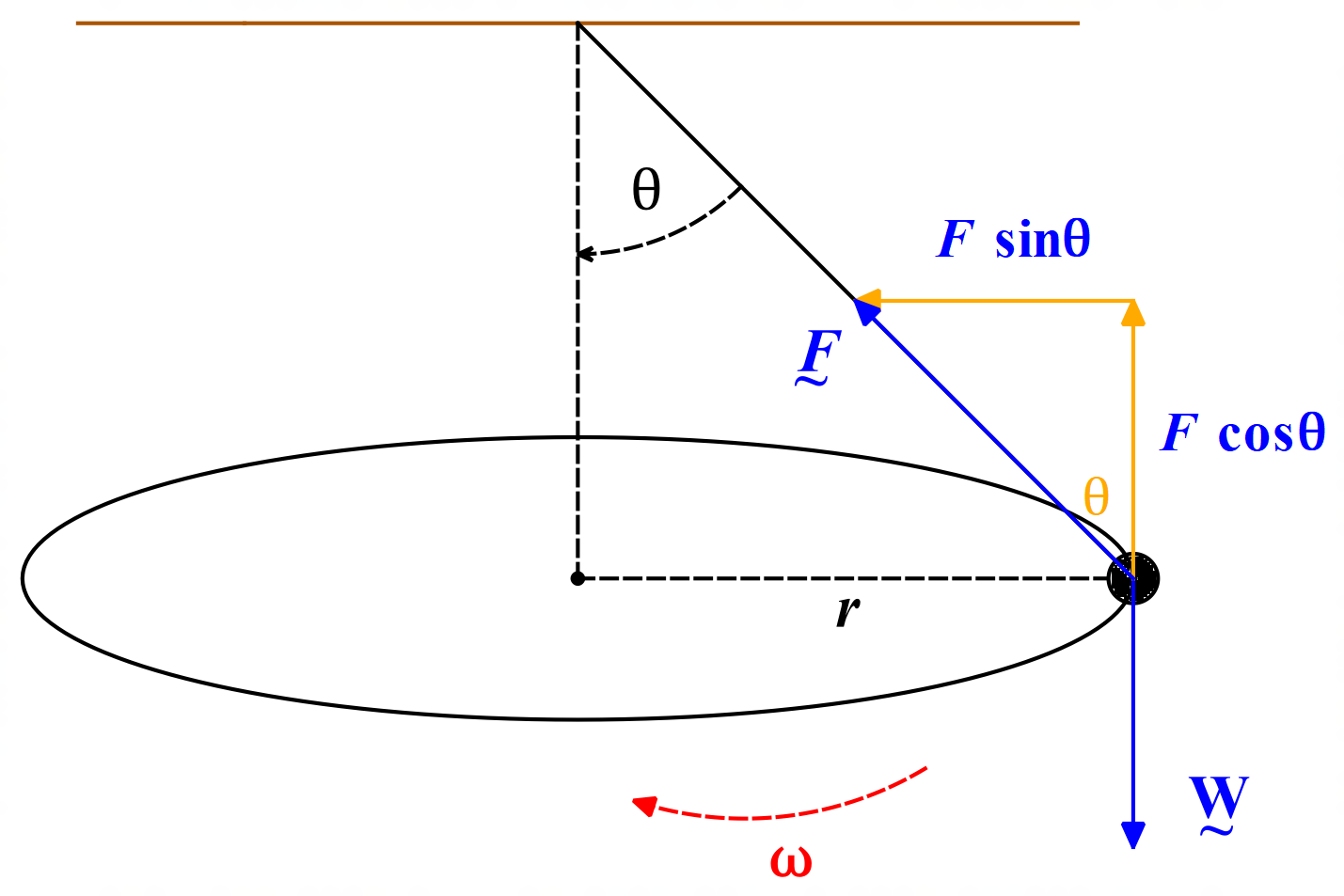
If you were asked to calculate the tension, **FT,** in the string in this example, it would be easy to do, since the tension in the rope is equal to the centripetal force. So, **FT = mv2 / r**. Given the mass and linear velocity of the ball and the length of the rope (the radius in this case), the tension in the rope could be determined.

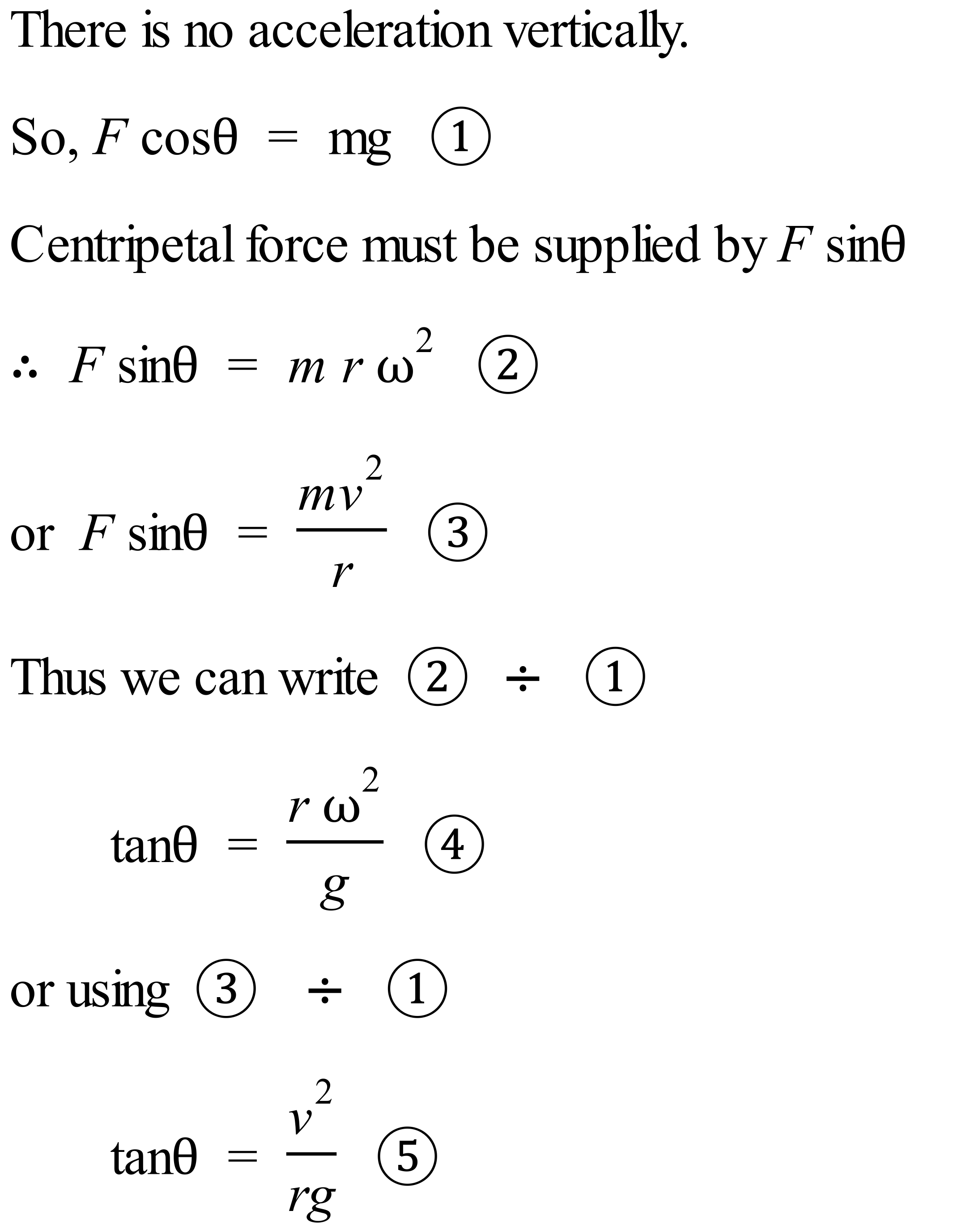
Note that if the rope was to break during the circular motion, the ball would fly off at a tangent to the circular path, by Newton’s 1st Law.

This [Kahn Academy video](https://www.khanacademy.org/science/physics/centripetal-force-and-gravitation/centripetal-forces/v/centripetal-force-problem-solving) provides a good explanation of this situation as well as good advice on solving numerical problems in UCM. I strongly advise you to watch the video. It also takes things further and has a look at a slightly more difficult situation. Well worth a look.

**Conical Pendulum**

The **conical pendulum** consists of a mass suspended by a light string. The mass moves in a horizontal circle of fixed radius **r**, as shown below.





The analysis above is the standard treatment of conical pendulum problems. This [Kahn Academy video](https://www.khanacademy.org/science/physics/centripetal-force-and-gravitation/centripetal-forces/v/mass-swinging-in-a-horizontal-circle) is worth a look at as an example of how to solve problems involving conical pendulums. Such problems are an excellent test of your understanding of vector resolution and UCM.

**Vertical Circle**

What if the mass on the string is swung in a **vertical circle**? An example would be a yo-yo doing an “around the world” maneuver. The physics & maths in this example are very useful in solving many problems including forces acting on pilots doing “loop the loops” and similar maneuvers. This [Kahn Academy video](https://www.khanacademy.org/science/physics/centripetal-force-and-gravitation/centripetal-forces/v/yo-yo-in-vertical-circle-example) provides an excellent example of the solution of such scenarios. In reality, there will always be a component of the weight force that is tangential to a vertical circular path. This component will change the speed of the mass (slowest at top, fastest at bottom) and the motion will not be UCM.

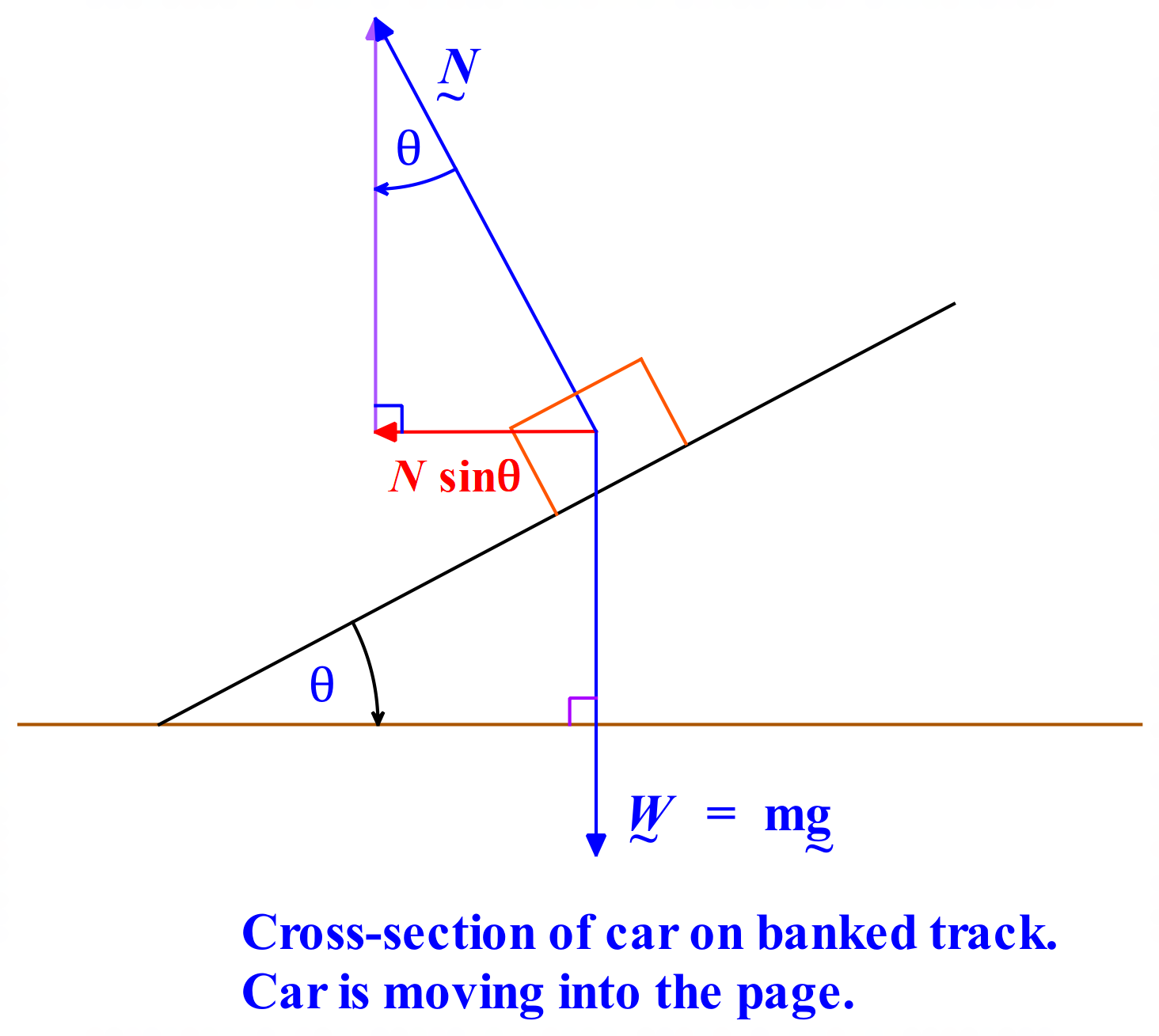
**Circular Motion Practical**

There is an excellent practical that can be done using a hollow glass tube with a string threaded through it. A mass carrier on the bottom of the string allows different masses to be added to change the tension in the string. A mass can be attached to the other end of the string to allow it to be swung in UCM. The relationships between several variables can be tested in this way. Hopefully, your teacher will do this prac with you. It’s usually a lot of fun. Safer to do outdoors, so people can spread out. [This website](https://www.wired.com/story/the-physics-of-swinging-a-mass-on-a-string-for-fun/) discusses this prac. I suggest you have a look.

**3. Objects on Banked Tracks**

The first example we examined in this section of the Module was of a car going around a horizontal circular bend on a level road. We saw that the centripetal force required to keep the car from slipping was supplied by the friction between the car tyres and the road surface. One technique that civil engineers use when constructing roads is to **bank the road**. This means that the outer edge of a curved road is raised above the inner edge to provide the necessary centripetal force to vehicles to ensure a safe path around the bend within a specified range of speeds.

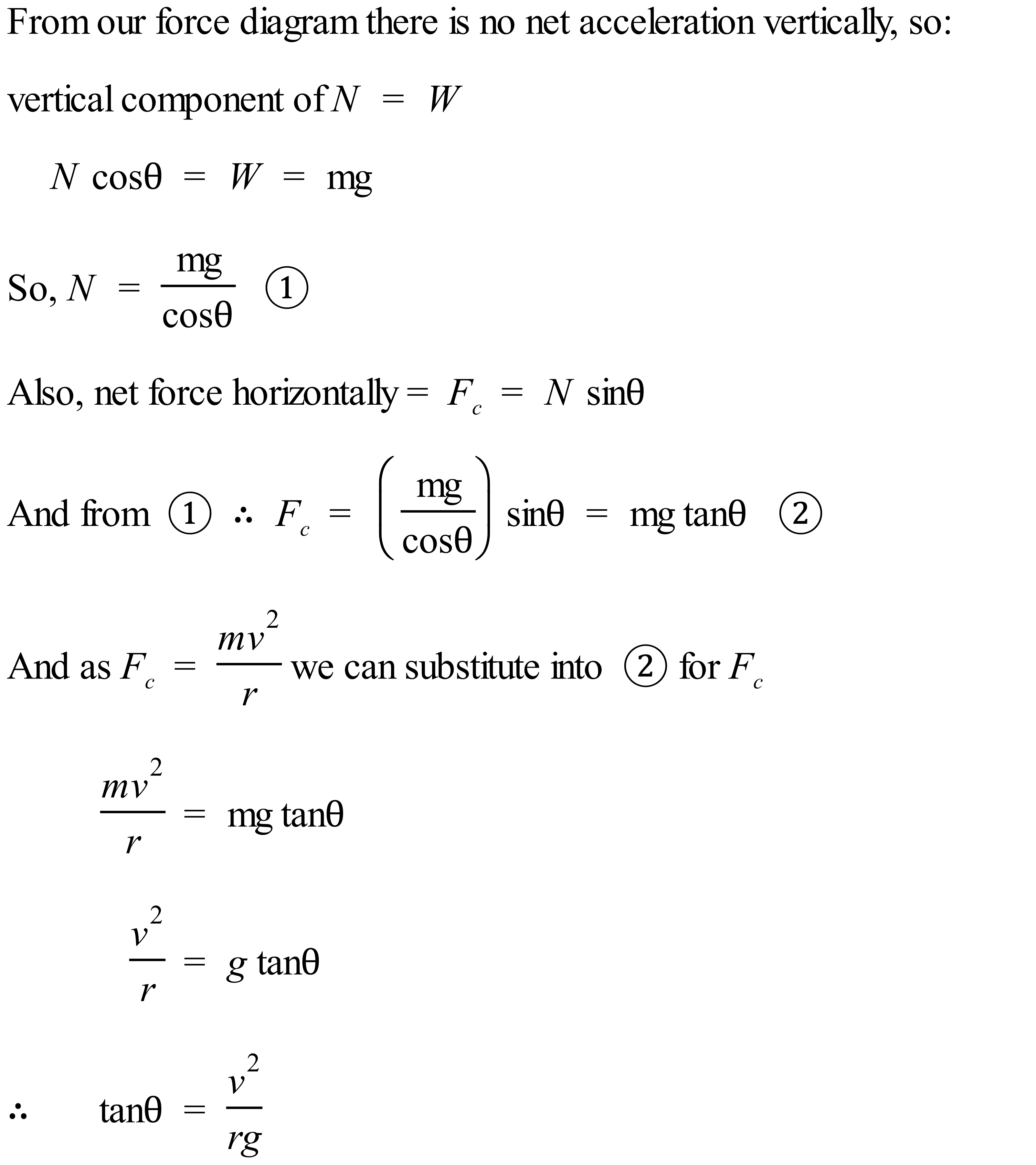
We shall now analyze the forces acting on a car moving around a bend on a banked road. Consider the diagram below. Ignore air resistance and friction force between tyres and road.



Study the force diagram above. Ensure you know how to create it. I have drawn a cross-section of the car as it moves around the banked track. The car is moving down into the page. The angle of bank is **θ**. I have drawn all forces acting on the car ignoring air resistance and friction between tyres and road. I have drawn the normal force, **N**, acting on the car perpendicular to the track; the weight force, **W**, acting downwards on the car; and the horizontal and vertical components of the normal force. Clearly, the only force acting in the horizontal direction, is the horizontal component of the normal force, **N sinθ**. This must provide the centripetal force that keeps the car in UCM around the bend.

Your initial thought might have been to resolve the weight vector parallel and perpendicular to the road - after all, that is what we did for all of those lovely inclined plane problems, remember? The difference is that in those problems we expected the object to accelerate parallel to the incline, so it made sense to have the vectors pointing parallel and perpendicular to the incline. Here, though, the acceleration is horizontal - toward the centre of the car's circular path - so it makes sense to resolve the vectors horizontally and vertically.

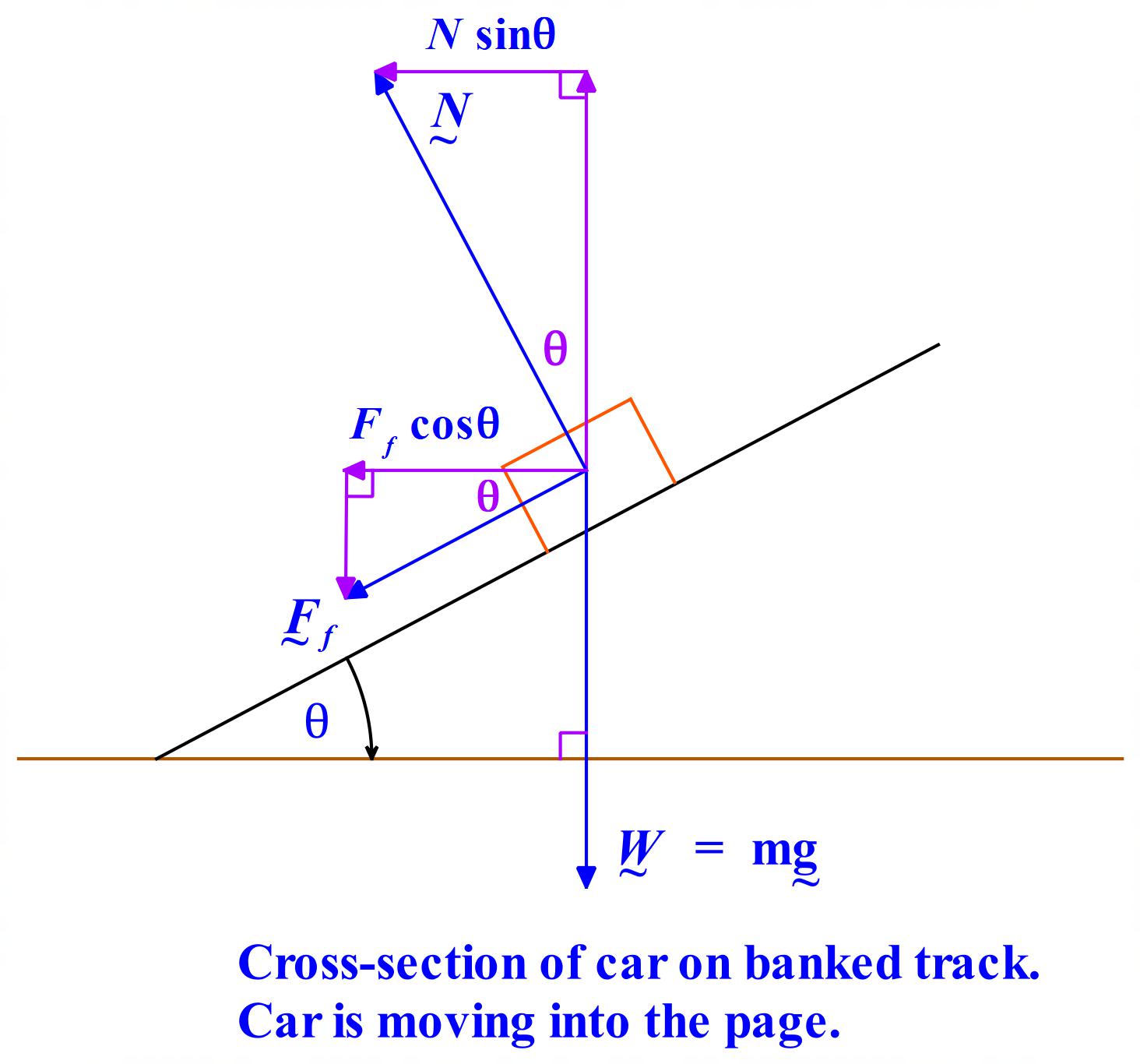
So, let’s determine the **angle of bank** that will allow a car to travel in a curve of radius **r** with constant speed **v** and require no friction force.



So, the angle of bank, **θ**, required to allow a car to travel in a curve of radius **r** with constant speed **v** and require no friction force can be determined using the formula above.

A banked curve is designed for one specific speed. Obviously, if we were initially given **θ**, we could determine the speed for which the banked curve was designed.

If the banked curve is icy so there is no friction force at all, then traveling at higher than design speed means the car will slide out, up, and over the edge and traveling at lower than design speed means the car will slide in, down, and off the bank. Fortunately, in Australia, icy roads are in the minority. For most of the roads we travel on, the curves are banked, and friction exits between the tyres and the road, so we have two sets of forces adding together to provide the necessary centripetal force to keep us on the road. We shall now have a look at this scenario.



Note that on the banked track the friction between the tyres and the road, **Ff**, acts parallel to the incline. Clearly, from our force diagram above, the net force acting horizontally is the sum of the horizontal components of the normal force and the friction force. This net force provides the centripetal force that keeps the car moving in a circular path around the bend. So, we have:

**Fnet = N sinθ + Ff cosθ = Fc**

[This web page](http://www.batesville.k12.in.us/physics/phynet/mechanics/circular%20motion/banked_no_friction.htm) provides a couple of example questions worth looking at. Ignore the third example question – it is in non-SI units.

**Energy & Work in UCM**

Recall the Work-Energy Theorem from Module 2 on Dynamics. It states that the work done on a body by the net force is always equal to the change in kinetic energy of the body. Let us consider what this means for objects executing UCM in a horizontal circle.

The kinetic energy (KE) of a mass in uniform circular motion depends only on the magnitude of its velocity. KE = ½ mv2 is the relevant formula. In UCM, the linear velocity of the mass remains constant in magnitude. Thus, the KE of the mass remains constant. **There is no change in KE and therefore no work done on the mass.**

This point could also be made by considering the definition of work:



where W = work done on object, F = net force acting on object along the line of motion and s = net displacement of object caused by the force along the line of motion. What this tells us is that only the component of the force in the direction of motion matters. In UCM, the centripetal force is always at right angles to the direction of motion of the mass. Therefore, there is no work done on the mass, because there is no component of force in the direction of motion.

As the UCM is occurring in a horizontal circle, the potential energy, PE, of the mass remains the same at all points along its path. **Thus, the total energy (KE + PE) of the mass remains constant in uniform circular motion.**

The earth in orbit around the sun provides an example here. If we approximate the earth’s orbit as UCM, the gravitational force between earth and the sun acts as the centripetal force that produces the motion. The centripetal force always acts towards the centre of the circle, but at each instant the linear velocity and therefore the displacement of the earth is at right angles to the centripetal force. Hence, from the arguments made previously, no work is done by the gravitational force and there is no change in potential energy, PE, or the kinetic energy, KE, of the Earth in its orbit. The total energy (PE + KE) is constant. Consequently, and very fortunately for us, the earth just keeps on orbiting.

**Rotation of Mechanical Systems and Applied Torque**

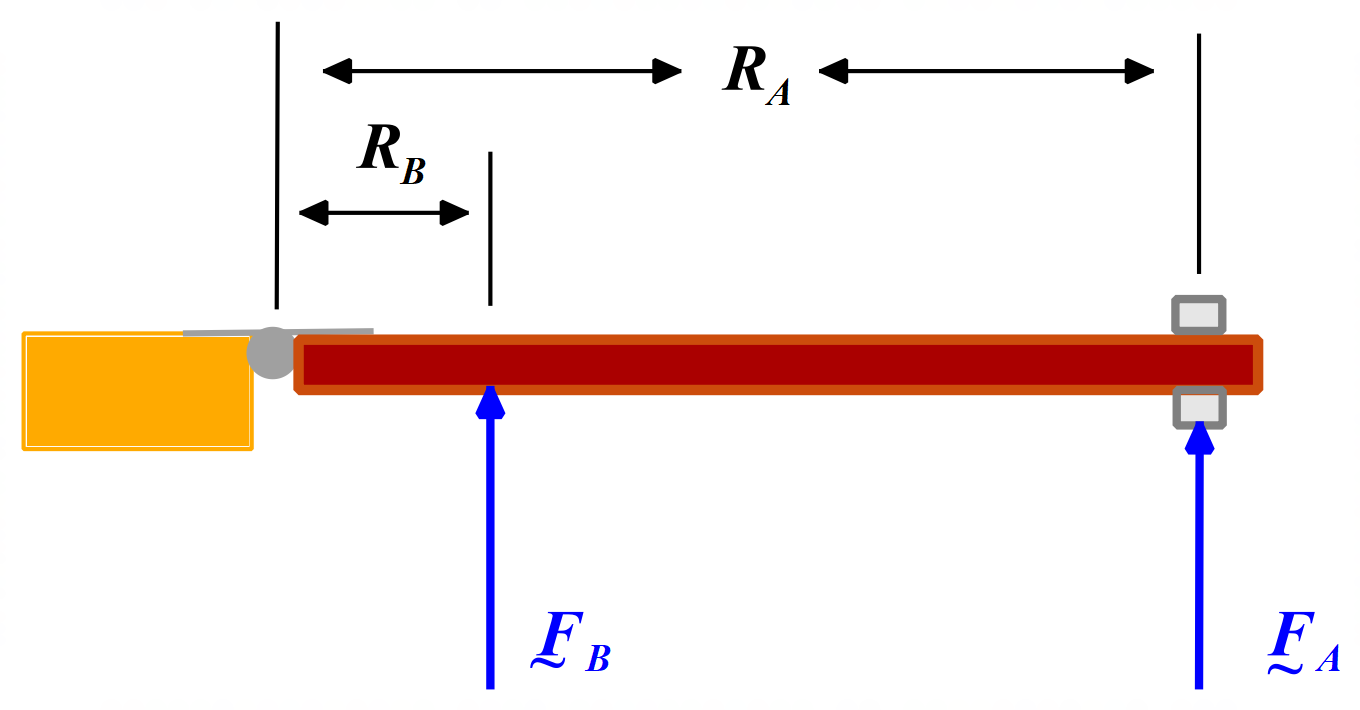
The study of UCM provides a very good introduction to the mathematical modelling of rotating systems. Our world is full of rotating systems, from the wheels on a car to the cogs in machines, to the spinning of high-speed turbines in a power station and so on. In Physics, objects such as wheels, cogs, turbines, etc are called rigid bodies. A rigid body is one whose component particles bear a fixed spatial relationship to each other; there is no relative movement among them. When a rigid body rotates about a fixed axis, all its component particles are in circular motion about that axis.

All real mechanical systems are composed of rigid bodies. The study of the rotation of rigid bodies in mechanical systems is called Rotational Mechanics and is comprised of Rotational Kinematics and Rotational Dynamics. We shall look very briefly at one aspect of rotational dynamics – the concept of **torque**.

In order to do that in a logical fashion, it is necessary to define the **angular acceleration**, ****. In UCM **** is constant. If **** changes with time, the angular speed is increasing or decreasing and therefore, we have an angular acceleration present. You do not need to remember this for the current course, but the average angular acceleration, **** = Δ**** / Δt.

To make an object start rotating about an axis clearly requires a force to give the object some angular acceleration. **Torque**, denoted by **** (the Greek letter tau), is the physical quantity that provides the required angular acceleration.

Study the diagram below, which depicts a top view looking down on a door. The hinge which provides the axis of rotation is at the left, attached to both the wall and the door. The door handle is at the right of the diagram.

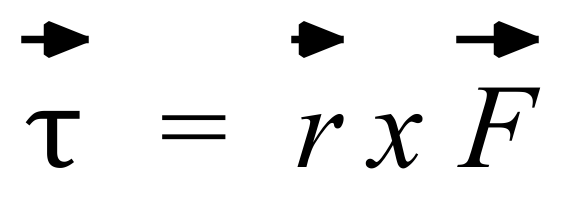


To open the door, you would normally apply a force, **FA**, as shown. The greater the magnitude of this force, the more quickly the door will open. If you applied the same magnitude force closer to the hinge, say **FB**, the door will not open so quickly. Although you applied the same sized force in both cases, the effect of the force is less in the second case. So, where the force is applied, as well as its magnitude and direction, affects how quickly the door opens.

Indeed, if only this one force acts, the angular acceleration of the door is proportional not only to the magnitude of the force but is also directly proportional to the perpendicular distance from the axis of rotation to the line along which the force acts. This distance is called the **lever arm,** or **moment arm,** of the force and is labelled **RA** and **RB** for the two forces showing in the above diagram. If **RA** is three times larger than **RB**, then the angular acceleration of the door will be three times as great, assuming the magnitudes of the forces are the same. This is why you would normally open the door by applying force to the handle rather than somewhere close to the hinge.

So, the angular acceleration is proportional to the product of the force times the lever arm. This product is called the **moment of the force about the axis**, or more commonly, the **torque**.

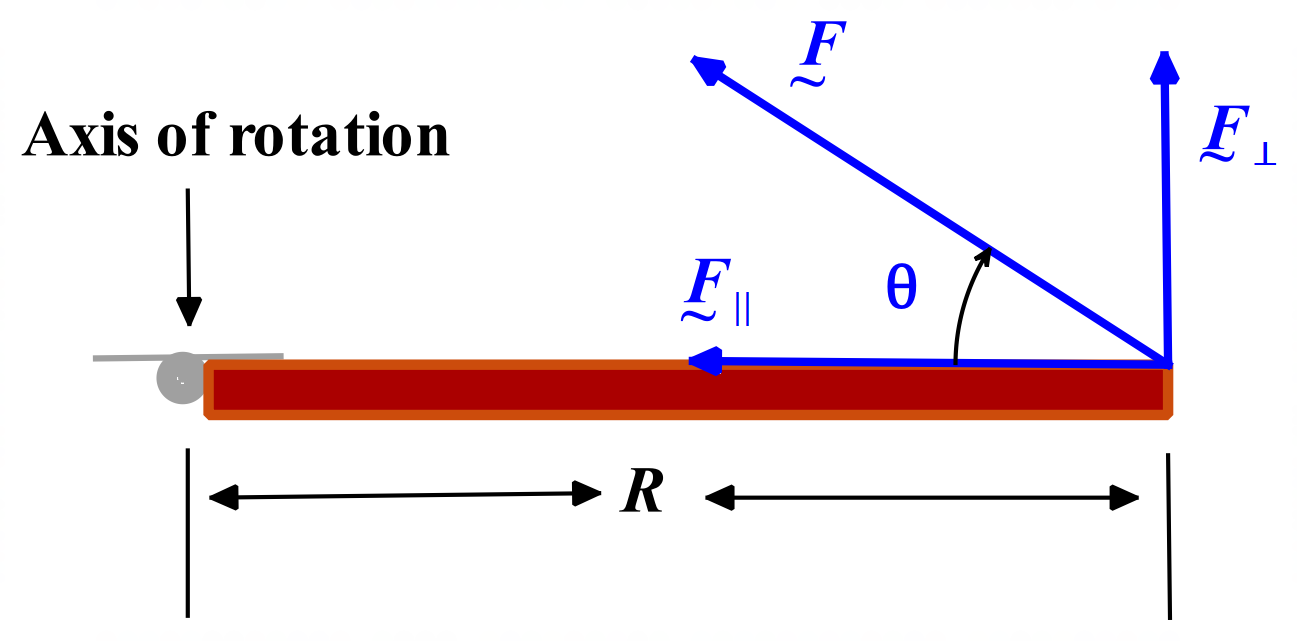
**Torque, then, is defined as the turning moment of a force.**  The torque about an axis of rotation is the product of the perpendicular distance of the axis from the line of action of the force and the component of the force in the plane perpendicular to the axis.



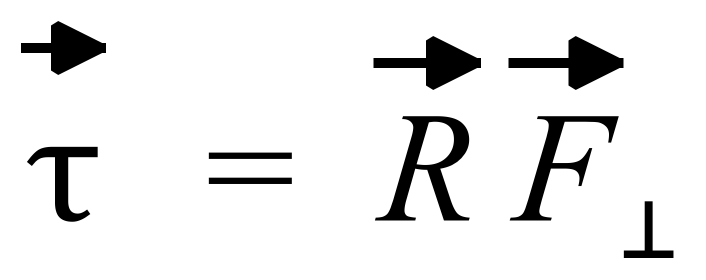
The formula above is the full mathematical definition of torque. I give it here for completeness. This is how you will learn it at university. The “x” in between the displacement and force vectors represents a “cross product” of the vectors. It is one type of vector multiplication and always produces another vector as its answer. Therefore, **** **is a vector quantity**. Its direction is given by the RH Rule mentioned earlier. It will always be perpendicular to both F and r. **For the Stage 6 course you can simply indicate whether the torque turns the object clockwise or anticlockwise.**

**The units of torque are Nm**. Although these units are the same as for work and energy, torque is not the same quantity as work or energy (scalar quantities), as can be seen very clearly from its definition. Do not ever write the units of torque as joules, which is only ever used as a unit of work or energy.

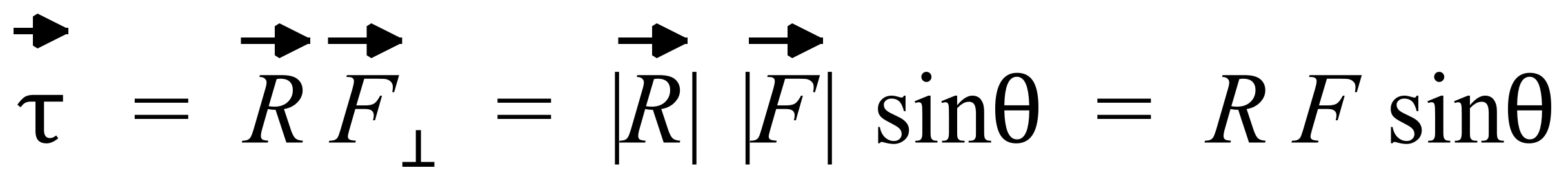
We shall now derive an expression for torque without the need for the cross product mentioned above. Consider the following diagram, which shows the same door as used previously. This time, however, we are applying a force at an angle, θ, to the very edge of the door. We proceed by resolving the force into components parallel and perpendicular to the line that connects the axis to the point of application of the force. This will produce the expression required by the syllabus. It is not the only derivation possible.



The component **F||** exerts no torque since it is directed at the rotation axis and therefore its moment arm is zero. Hence, the torque will be equal to **F⊥** times the distance **R** from the axis to the point of application of the force:



Clearly, from the diagram **F⊥** = **F** sin θ and so we have finally:



And we usually just use a lowercase **r** for the distance.

Note that **θ** is the angle between the directions of the vector **F** and the distance **R** (radial line from the axis to the point where **F** acts).

Also note that there is an alternative way to derive the formula above. Instead of resolving the applied force into components, we could leave the force as it is and determine the perpendicular distance from the axis of rotation to the line of action of the force. It arrives at the same result. I have provided this alternative derivation in Appendix B at the end of these notes for anyone who is interested. In physics, it is always good to be able to see more than one way to attack a problem.

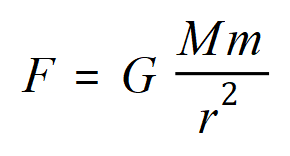
You might like to have a look at this [Khan Academy video](https://www.khanacademy.org/science/ap-physics-1/ap-torque-angular-momentum/torque-and-equilibrium-ap/v/finding-torque-for-angled-forces), which provides a couple of example torque problems.

### **MOTION IN GRAVITATIONAL FIELDS**

**Inquiry Question:** How does the force of gravity determine the motion of planets and satellites?

**Gravity and the Gravitational Field**

In 1687 Isaac Newton published his **Principia Mathematica**, in which he explained his three **Laws of Motion** and his **Law of Universal Gravitation**.  The Law of Universal Gravitation states:



where F = the force of gravitational attraction between two masses, **M** and **m** a distance **r** apart and G = the Universal Gravitational Constant = 6.673 x 10-11 Nm2kg-2. In words this law can be expressed as:

***The force of attraction between any two bodies in the universe is proportional to the product of their masses and inversely proportional to the square of their distance apart.***

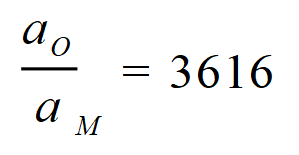
Newton’s law suggests that every mass in the universe, no matter how small, has its own gravitational field surrounding it. The larger the mass, the stronger the gravitational field around it. This field is a region of influence in which another mass would experience a force due to the presence of the first mass.

**Derivation of the Gravitational Force Law**

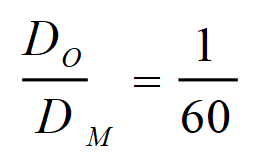
How Newton arrived at the above formula is an interesting study in inductive and deductive reasoning. It was a derivation based on empirical observations. Newton was convinced that the same gravity force that attracted earthbound objects towards the centre of the Earth and caused them to fall when dropped from a height, was responsible for keeping the Moon in orbit. He reasoned that if it were not so, the Moon’s inertia would cause it to continue to move in a straight line (according to his 1st Law of Motion) and it would disappear from orbit.

Newton used his 2nd & 3rd Laws to infer that mass had to be involved in any equation for gravitational force. If the Earth applied a gravitational force that accelerated the Moon toward it, the 2nd Law said that the force must be of the form **F = m aM** , where **m** was the mass of the moon and **aM** its acceleration. The 3rd Law implied that there must be an equal but oppositely directed force that the moon applies on the Earth. That force could be expressed as **F = M aE**, where **M** was the mass of the Earth and **aE**, the acceleration of the Earth because of the force. So, both masses must be involved in the gravitational force equation.

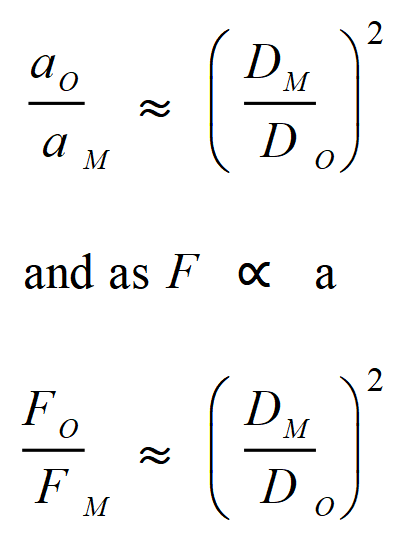
Newton used his centripetal acceleration equation and the relevant data known at that time to calculate a ratio of the acceleration due to gravity of an object on the surface of the Earth to the acceleration of the Moon due to Earth’s gravitational force acting on it.



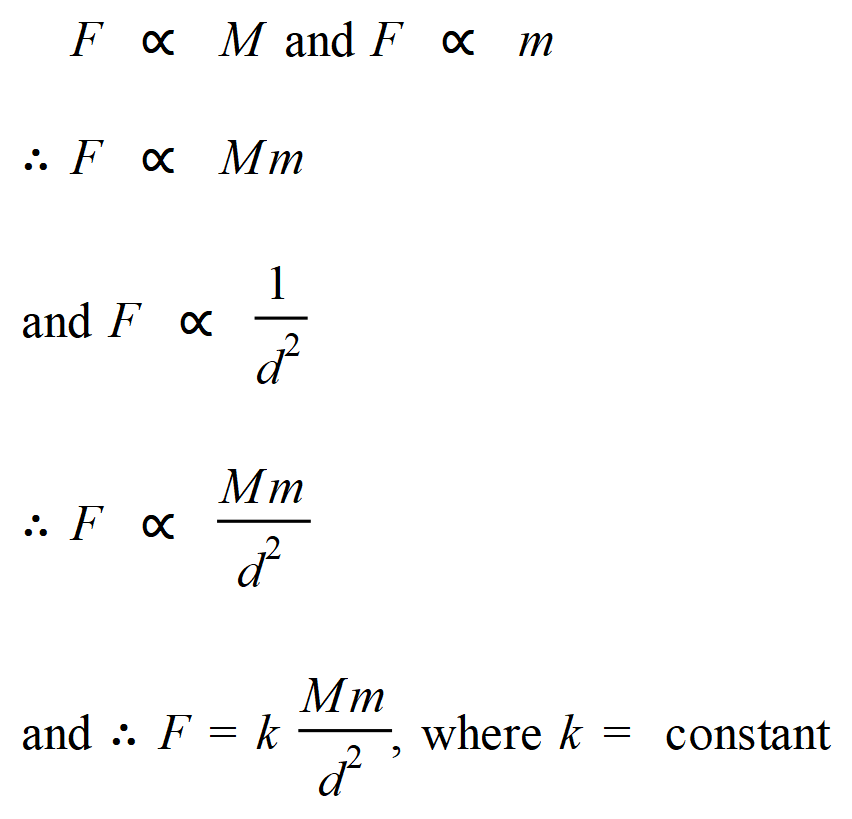
Newton also reasoned that the force would diminish at increasing distance, so he determined the ratio of the distance from the centre of the earth of an object on the surface of the Earth, to the distance of the Moon from the centre of the Earth.



Newton noticed that



So, putting all the pieces together, Newton had derived that



Newton did not have the relevant data to calculate an accurate value of the constant. The value was determined as a byproduct of a meticulous and elegant experiment performed by [Henry Cavendish](https://www.aps.org/publications/apsnews/200806/physicshistory.cfm) during 1797-1798, in which his aim was to determine the specific gravity (density) of the earth. Although neither the symbol G, nor the term “universal gravitational constant” appear in Cavendish’s paper, his experiment effectively measured its value. Physicists today credit Cavendish with the first accurate measurement of the value of G.

**Revision of the Definitions of Mass and Weight:**

The **mass** of an object is a measure of the amount of matter contained in the object.  Mass is a scalar quantity.

The **weight** of an object is the force due to a gravitational field acting on the object.  Weight is a vector quantity.

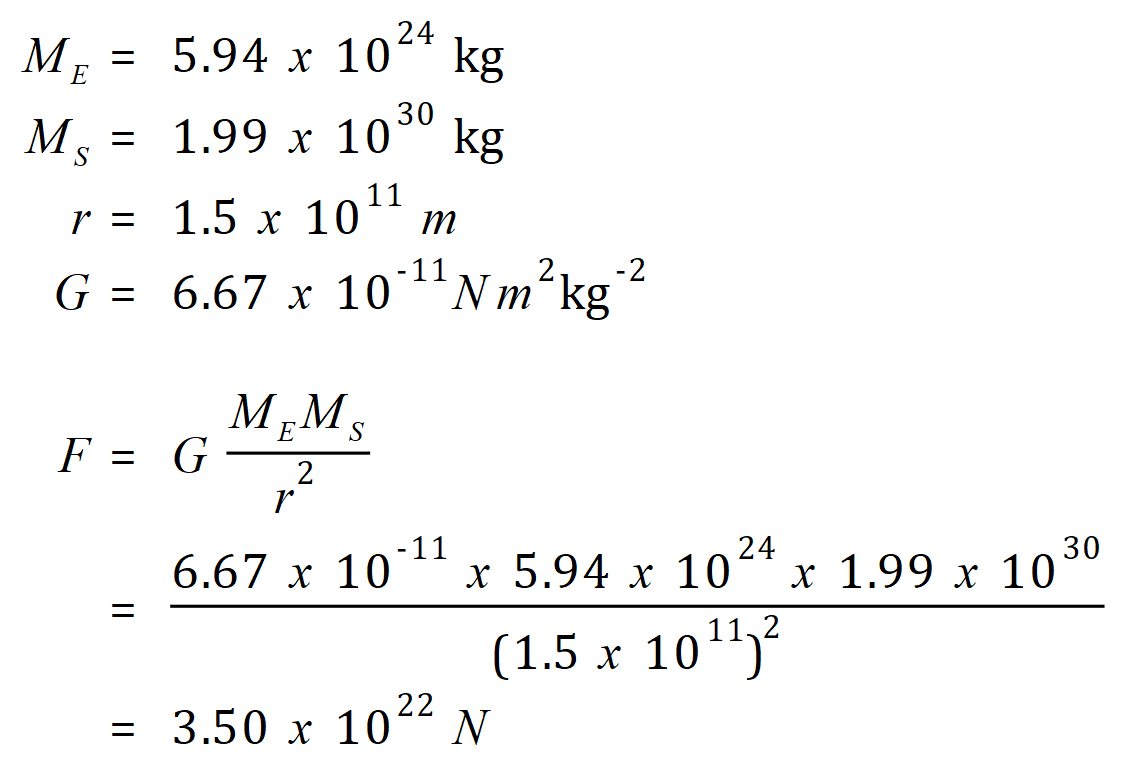
The weight, W, of an object is given by Newton’s 2nd Law as:



where **m** is the mass of the object and **g** is the acceleration due to gravity (9.8 ms-2 close to the earth’s surface).

**Exercise:** Determine the gravitational force acting between the Sun and the Earth, given that the mass of the Sun is 1.99 x 1030 kg, the mass of the Earth is 5.94 x 1024 kg and the mean Sun-Earth distance is 1.5 x 108 km.

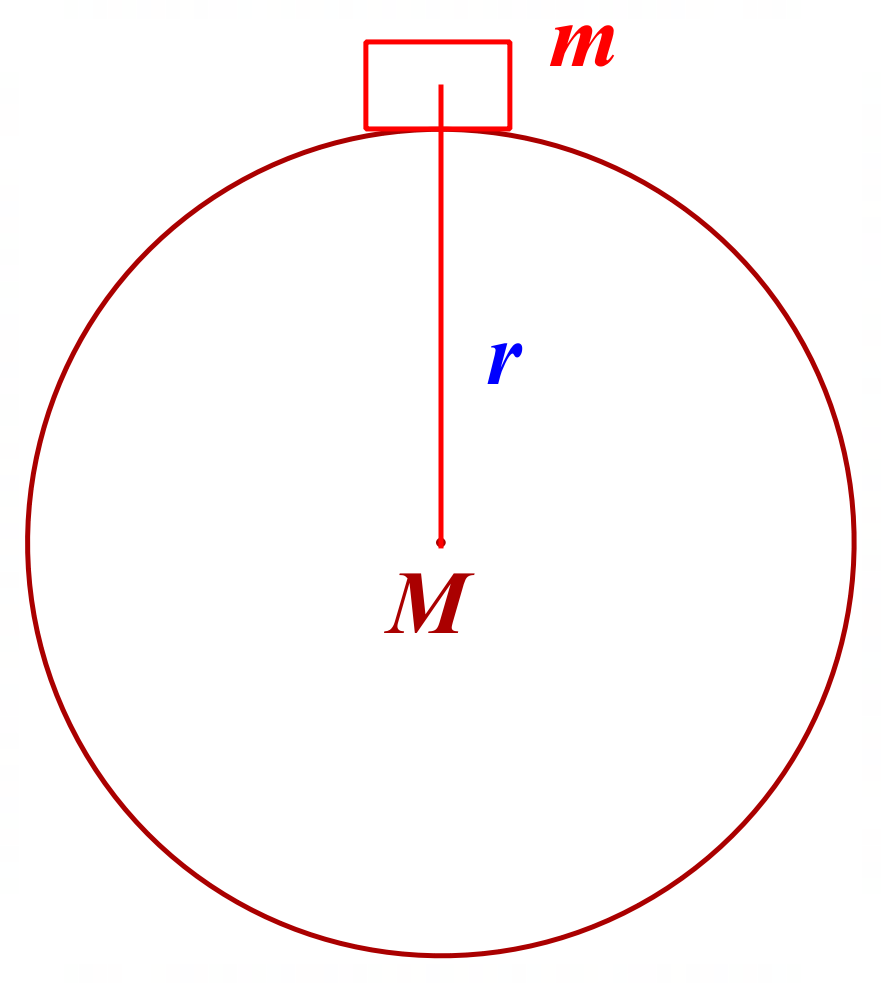
**Solution:**



Therefore, the gravitational force acting between the Sun and the Earth is 3.50 x 1022 N attractive force, acting along the line joining the centres of the two bodies.

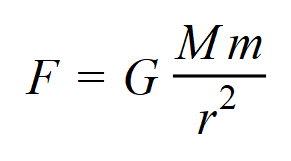
**Factors that Affect the Gravitational Field Strength**

Consider a mass, **m**, sitting on the surface of a planet of mass, **M**, and radius **r**, as shown below.



The diagram is not to scale obviously, so we can see the smaller mass.

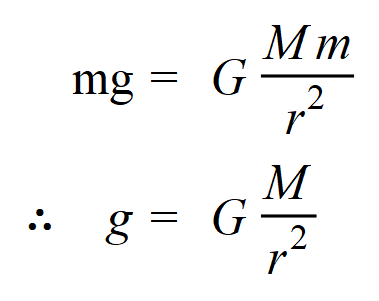
The gravitational force of attraction between the two masses is given by:



The weight force acting on the smaller mass can be written using Newton’s 2nd Law as:



And as the weight force acting on **m**, must be equal to the gravitational force of attraction on **m** toward the centre of the planet, we can write:



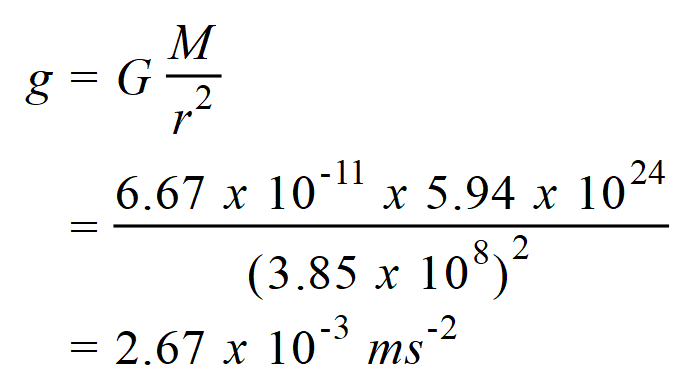
**Clearly, the gravitational field strength around any planet is directly proportional to the mass of the planet and inversely proportional to the square of the radius of the planet.** Also, the formula above allows us to calculate the strength of the gravitational field of a planet at any distance from the centre of the planet. The further away from the planet the smaller the gravitational field strength.

**Exercise:** Use the equation for the gravitational field strength above to calculate the acceleration due to gravity on other planets.  Use the mass and radius data from this [webpage](http://www.aerospaceweb.org/question/astronomy/q0227.shtml) and check your values of acceleration due to gravity against those in the table.

**Exercise:** Predict the gravitational field strength at a distance of 3.85 x 108 m from the centre of the Earth. Given that this distance is the time averaged distance between the centres of the Earth and Moon, determine the force of gravitational attraction acting on the Moon due to the presence of the Earth. Mass of the Moon is 7.36 x 1022 kg. Other data is as previously given.

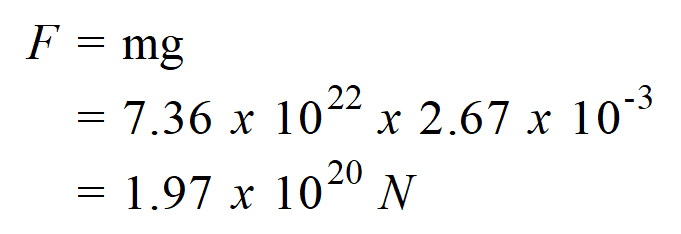
**Solution:**

Gravitational field strength is given by:



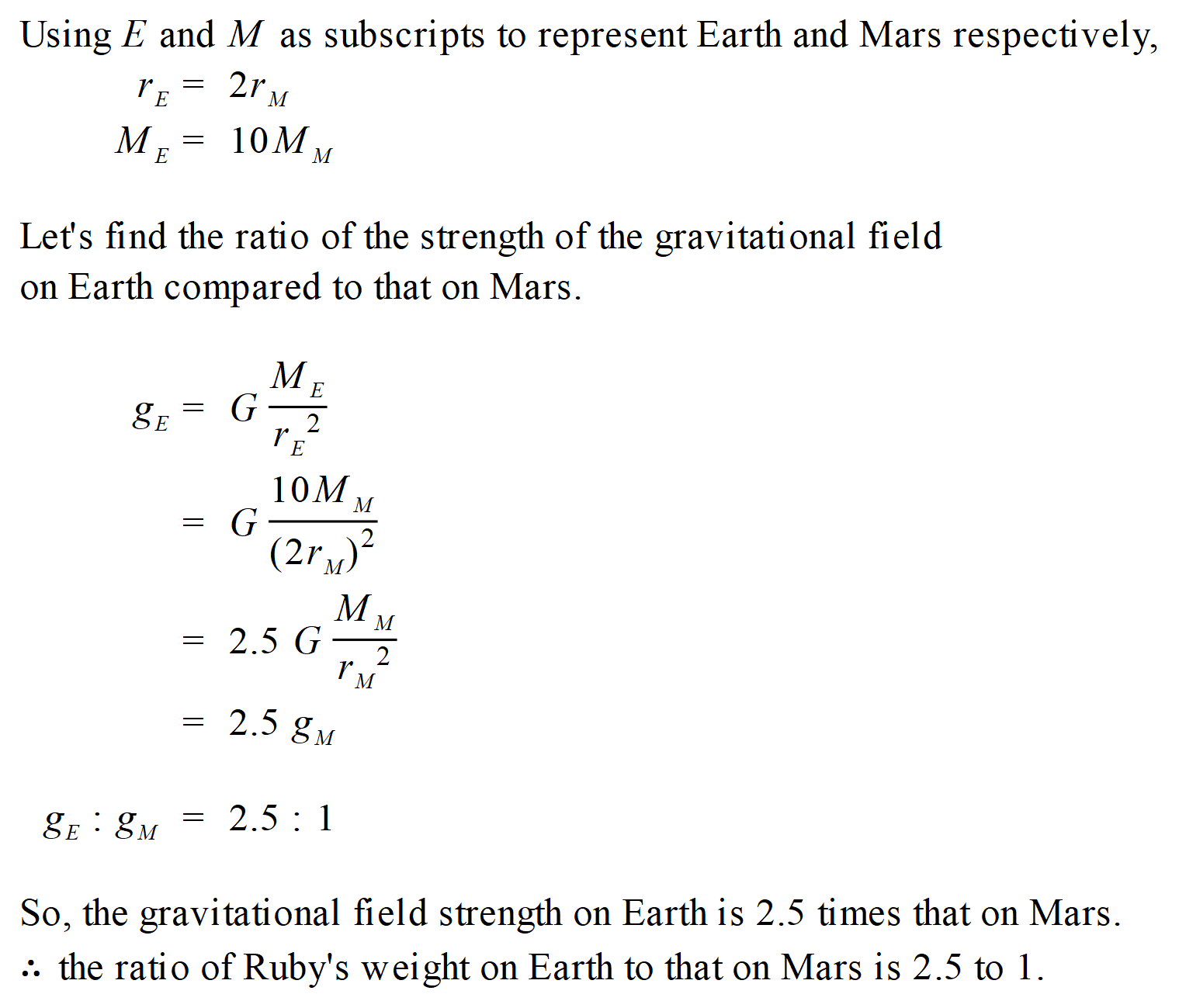
So, the gravitational field strength at a distance of 3.85 x 108 m from the centre of the Earth is 2.67 x 10-3 ms-2.

Therefore, the force of gravitational attraction acting on the Moon due to the presence of the Earth can be obtained using Newton’s 2nd Law, since we now know the acceleration due to Earth’s gravity acting on the moon.



**Exercise:** Mars has a radius which is half that of Earth and has a mass one-tenth that of Earth. Astronaut Ruby has a mass of 85 kg. (a) Determine the ratio of Ruby’s mass on Earth to her mass on Mars. (b) Determine the ratio of Ruby’s weight on Earth to her weight on Mars.

**Solution:**

1. Mass does not change due to location. The ratio of Ruby’s mass on Earth to her mass on Mars is 1:1.
2. 

**Kepler’s Laws of Planetary Motion**

The history of the development of our modern model of the solar system is a fascinating study. One important part of that history involved Johannes Kepler (1571-1630). In his late twenties, Kepler took a job with Tycho Brahe (1546-1601), a wealthy astronomer (with a bronze nose – I leave you to look up the details of how he came to have a bronze nose). Brahe had collected a huge quantity of extremely accurate astronomical observations in the hope that they would support the geocentric model of the solar system (earth at centre). Kepler was employed to analyze these data.

Kepler struggled for many years to show that Brahe’s data indicated that planets had perfectly circular orbits. At that time the circle was believed to be the perfect shape by divine order and it was assumed that the orbits of planets would be circles.

Eventually, however, Kepler noticed that an imaginary line drawn from a planet to the Sun swept out an equal area of space in equal times, regardless of where the planet was in its orbit. If you draw a triangle out from the Sun to a planet’s position at one point in time and its position at a fixed time later—say, 5 hours, or 2 days—the area of that triangle is always the same, anywhere in the orbit. For all these triangles to have the same area, the planet must move more quickly when it is near the Sun, but more slowly when it is farthest from the Sun.

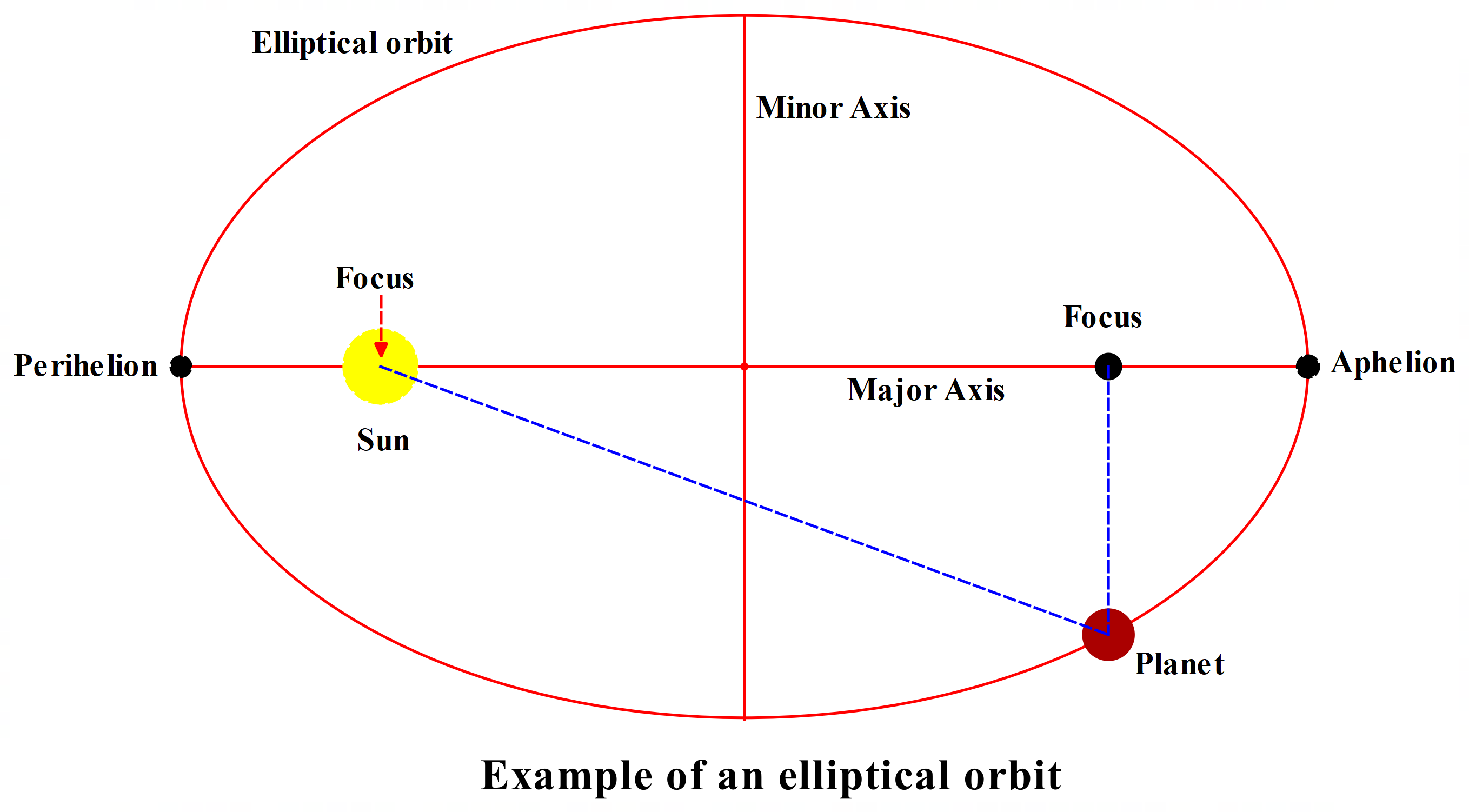
This discovery (which became Kepler’s second law of orbital motion) led to the realization of what became Kepler’s first law: that the planets move in an ellipse (a squashed circle) with the Sun at one focus point, offset from the centre.

Kepler found that Brahe’s data also indicated that there is a precise mathematical relationship between a planet’s distance from the Sun and the amount of time it takes revolve around the Sun. This became Kepler’s third law. It was this law that inspired Newton, who came up with three laws of his own to explain why the planets move as they do.

By the time Kepler had finished his analysis of Brahe’s data, he had effectively given extremely strong support to the heliocentric model of the solar system (the sun at the centre with the planets revolving around it).

Stated in order, Kepler’s three Laws of Planetary Motion are as follows.

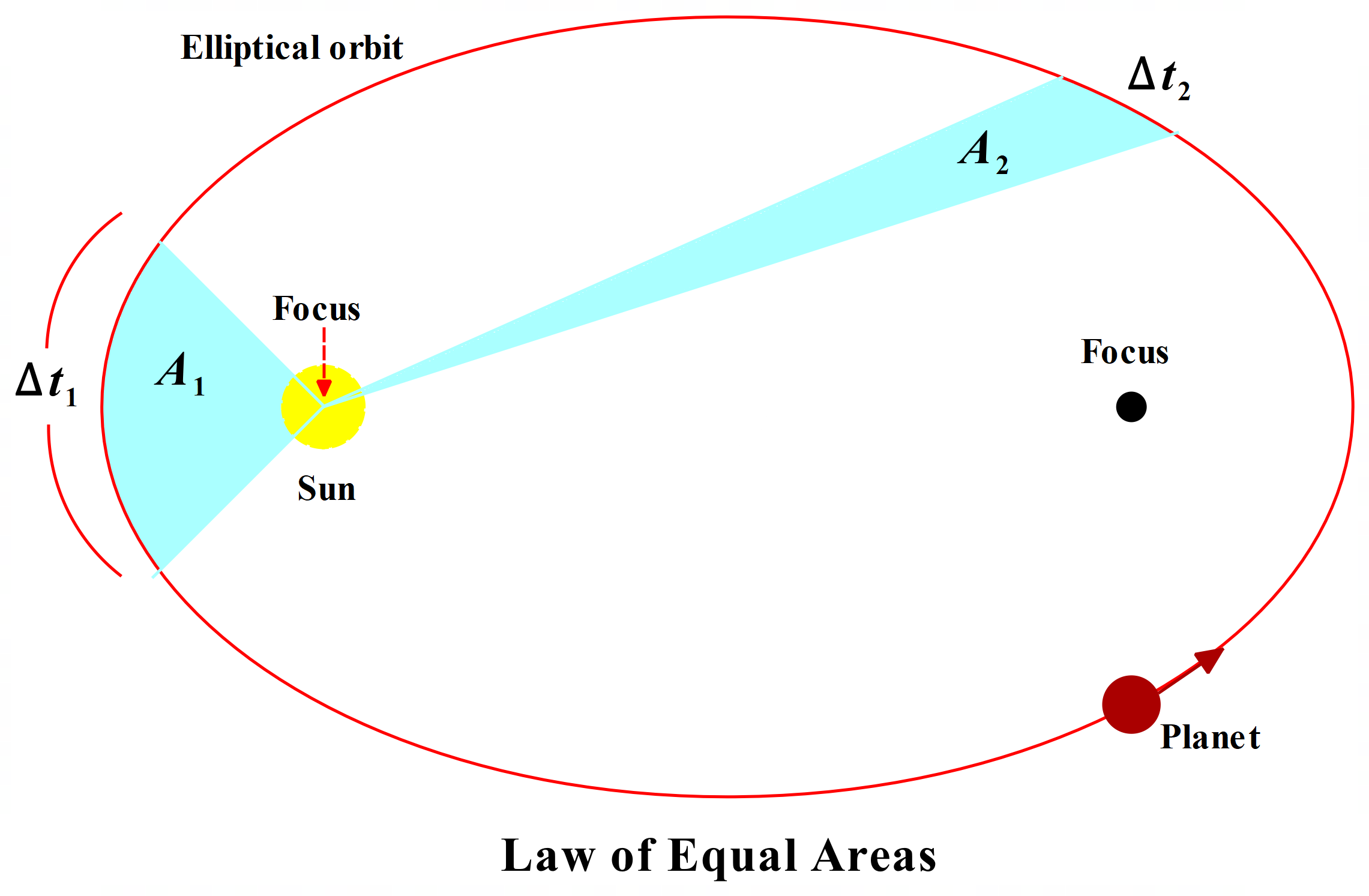
**First Law (Law of Ellipses):** The planets orbit the Sun in elliptical orbits, with the Sun as one common focus.



Note that most planetary orbits in our solar system are quite close to circular. The **orbital eccentricity** of an astronomical object is a dimensionless parameter that determines the amount by which its orbit around another body deviates from a perfect circle. A value of 0 is a circular orbit, values between 0 and 1 form an elliptic orbit, 1 is a parabolic escape orbit, and greater than 1 is a hyperbolic orbit. We will say a bit more about orbits later. The eccentricity of the Earth's orbit is currently about 0.0167; the Earth's orbit is nearly circular. Venus and Neptune have even lower eccentricities.  Mercury has the greatest orbital eccentricity of any planet in the Solar System (*e* = 0.2056). Such eccentricity is sufficient for Mercury to receive twice as much solar irradiation at perihelion compared to aphelion. Before its demotion from planet status in 2006, Pluto was the planet with the most eccentric orbit (*e* = 0.248).

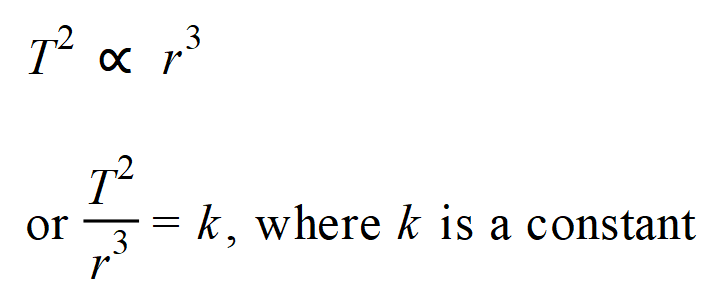
**Second Law (Law of Equal Areas):** The line between a planet and the Sun (the radius vector) sweeps out equal areas in equal periods of time.

This can be seen in the following diagram. Time intervals Δt1 & Δt2 are equal and therefore area A1 equals area A2. As mentioned previously, for these areas to be equal, the planet must move more quickly when it is near the Sun, but more slowly as it gets father from the Sun. So, a planet is moving fastest at perihelion and slowest at aphelion.

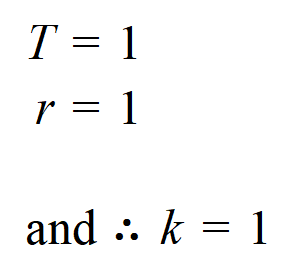


**Third Law (Law of Periods or the Harmonic Law):** The square of a planet’s period, **T**, is directly proportional to the cube of its average distance from the Sun, **r**.

Mathematically, this can be expressed as:



We can force k = 1 by measuring T in Earth years and r in astronomical units. Then we know that for the Earth, **T** = 1 year and **r** = 1 AU and in the equation above we have:



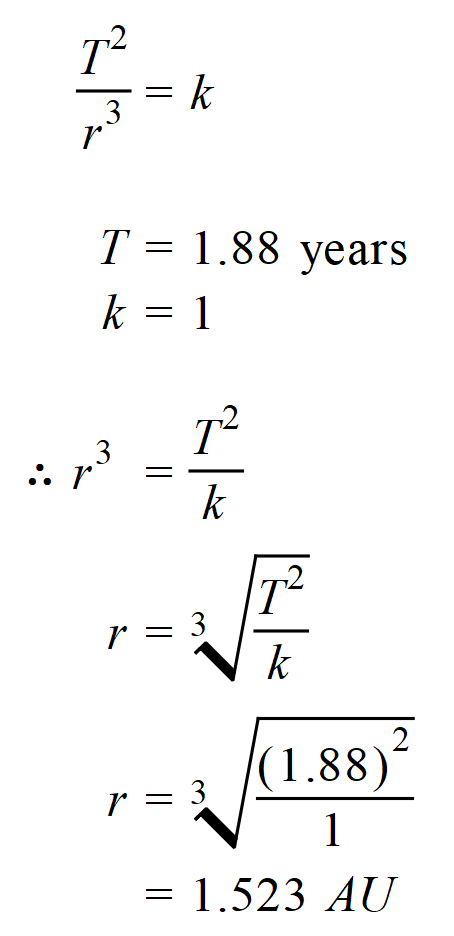
This law implies that the further a planet is from the Sun, the longer the planet’s period and the slower the planet moves around the Sun.

**Note: 1 AU = 1.5 x 108 km**

**Exercise:** Calculate the average distance from Mars to the Sun, given that Mars takes 1.88 Earth years to orbit the Sun.

**Solution:**

From Kepler’s 3rd Law we have the following:



The average distance from Mars to the Sun is 1.52 AU which is about 228.49 million km.

**Exercise:** Calculate the average time for a satellite to orbit the Earth at an altitude of 1000 km from the surface, given the following information: radius of Earth = 6.37 x 106 m; radius of the Moon’s orbit around Earth = 3.84 x 108 m; period of the Moon’s orbit around Earth = 2.36 x 106 s.

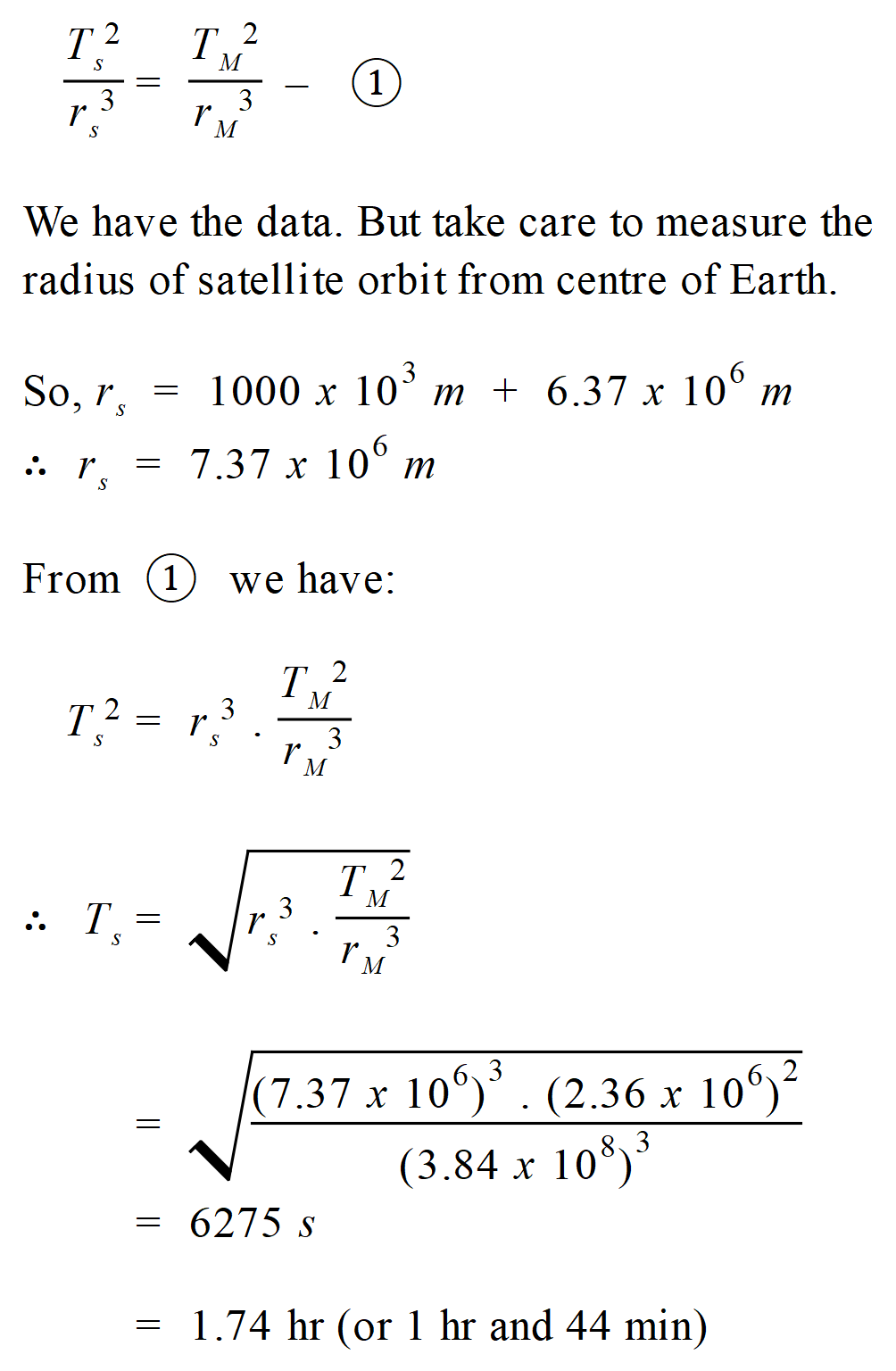
**Solution:**

When the data in the question is radii and periods of orbits, we are probably looking at using Kepler’s Law of Orbits to solve the problem.

**Remember, what Kepler’s 3rd Law is really about. For smaller masses orbiting a much larger central mass, the ratio of the square of the period of orbit to the cube of the radius of orbit is constant.**

As a side note – where the smaller mass is an appreciable fraction of the central mass, we must use an adjusted form of Kepler’s 3rd Law equation (called Newton’s form of Kepler’s 3rd Law). This is beyond the scope of the current module.

So, since both the satellite in question and the Moon are in orbit around the Earth, we can write:



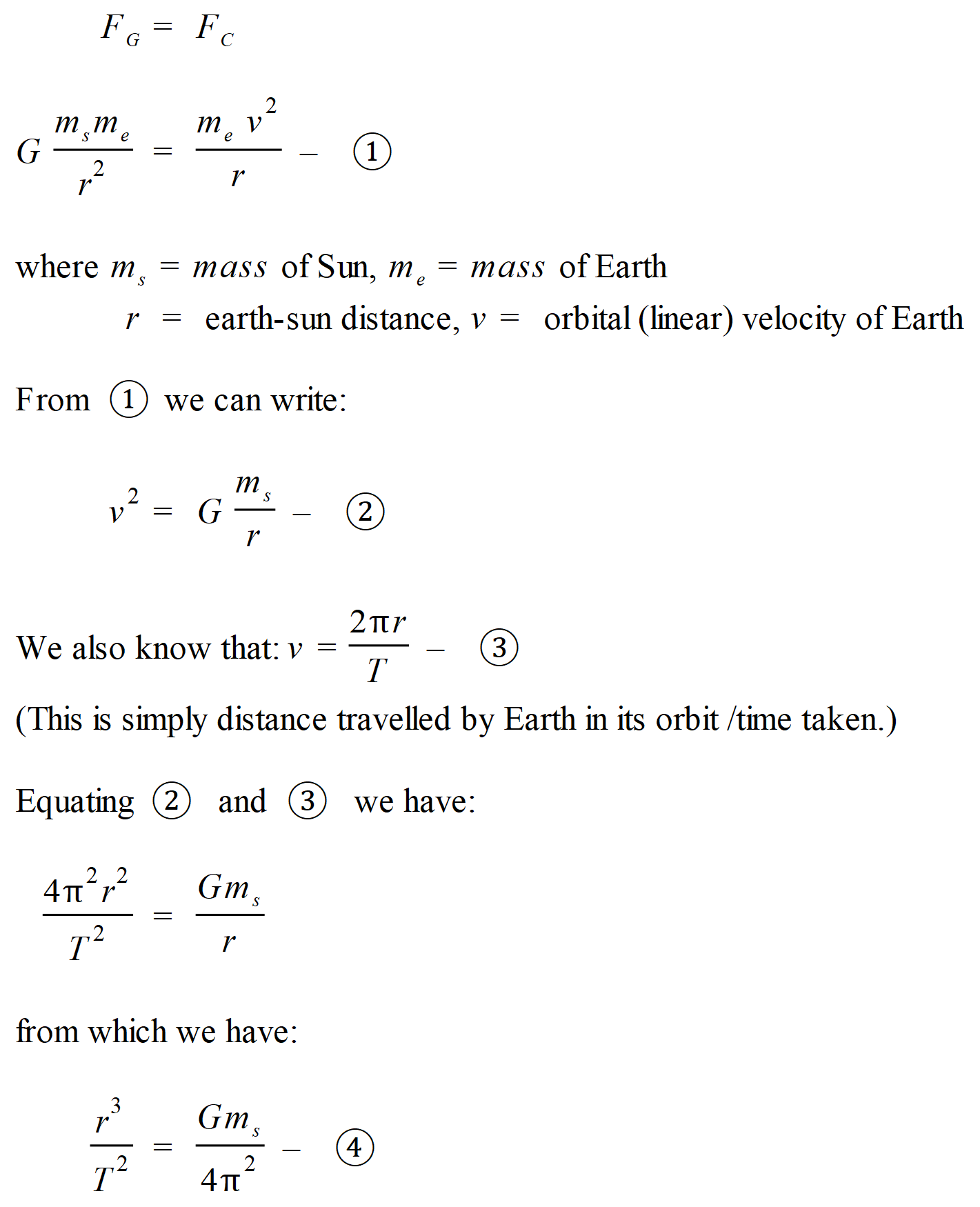
So, the satellite has an orbital period of 6.28 x 103 s.

Note that you will sometimes see Kepler’s 3rd Law expressed as above in equation 1, or as some rearrangement of this.

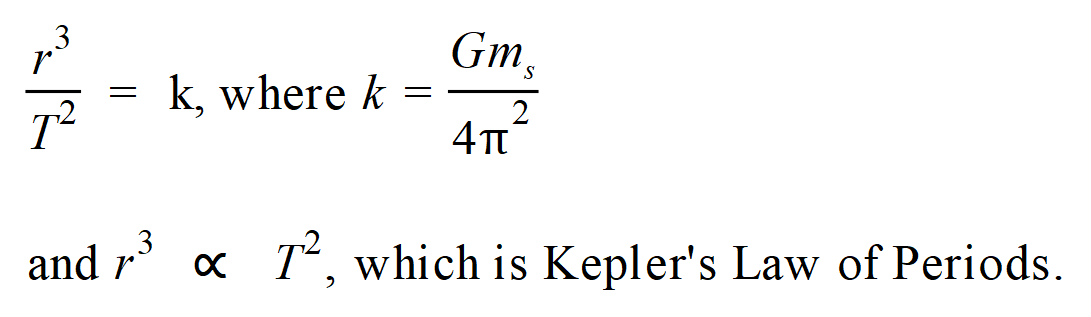
Note that calculations like those above involve quite a few steps. Be careful when using your calculators. It is very easy to make a mistake with these – many people forget to cube the radius or square the period for instance.

**Newton’s Further Work**

For a two-body system such as the Sun and the Earth, an equilibrium exists. The gravitational force of attraction between the two bodies, **FG**, is balanced by the centripetal force, **FC**.



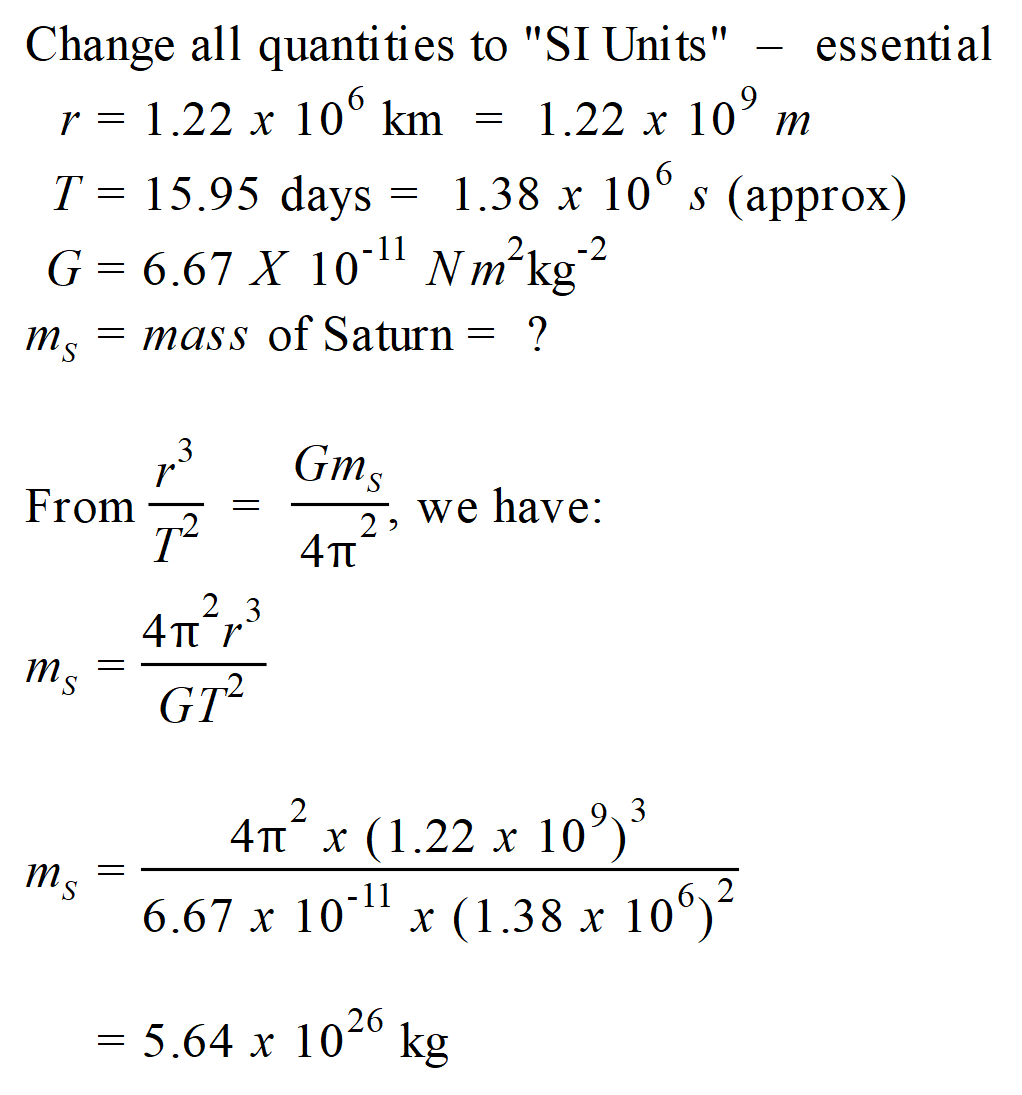
The above is Newton’s derivation of Kepler’s 3rd Law. Clearly, in equation 4, all terms on the RHS are constants, so we have that:



The value of k above is the same for all planets in our solar system. It will be different in another solar system because the sun in that system will almost certainly have a different mass. It is also worth noting that you can use Newton’s circular motion mechanics to derive Kepler’s Law of Periods by using the centripetal force equation in terms of , the angular velocity of Earth around the sun, instead of using the equation relating centripetal force to the linear velocity of Earth around the sun.

**Exercise:** Determine the mass of Saturn, **mS**, given that one of its moons, Titan, orbits it at a mean distance of 1.22 x 106 km every 15.95 days.

**Solution:**



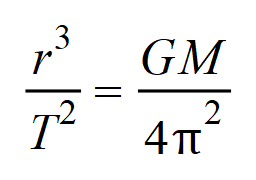
So, the mass of Saturn is 5.64 x 1026 kg.

**Orbital Velocity**

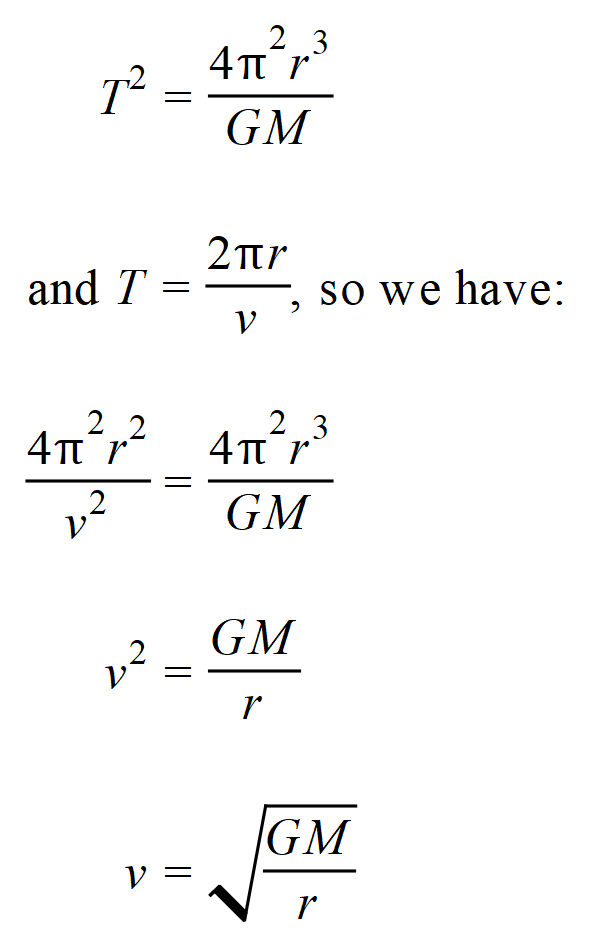
**Orbital velocity** is the velocity needed to achieve balance between gravity's pull on a satellite and the inertia of the satellite's motion – the satellite's tendency to keep going in a straight line.

Without gravity, the satellite's inertia would carry it off into space. Even with gravity, if the intended satellite goes too fast, it will eventually fly away. If the satellite goes too slowly, gravity will pull it back to the planet’s surface. At the correct orbital velocity, gravity exactly balances the satellite's inertia, pulling down toward the planet’s centre, just enough to keep the path of the satellite curving like the planet's curved surface, rather than flying off in a straight line.

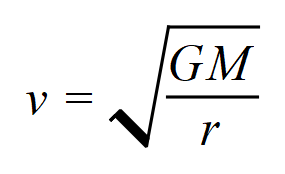
To achieve and maintain a stable orbit around a planet, a satellite must have the required orbital velocity.  If we assume the orbit of the satellite around the planet is circular, we can use Kepler’s Third Law to obtain an equation for the orbital velocity of the satellite.  Starting with the Law of Periods equation:



where **T** = period of satellite around the planet, r = distance from centre of the planet to the satellite, **M** = mass of planet and **G** = gravitational constant. Let us rearrange the equation with **T2** as the subject and then let us substitute for **T** in terms of the linear (orbital) velocity of the satellite, **v**, from circular motion theory.



Thus, the **orbital velocity** of a planet or satellite is given by:



where **T** = period of satellite around the planet, r = distance from centre of the planet to the satellite, **M** = mass of planet and **G** = gravitational constant. The direction of the orbital velocity is a tangent to the orbit at the point in question.

Take care when applying this formula. You will often be given the altitude of the satellite above the surface of the planet. Remember that the r in the formula is the centre to centre distance. So, often, you will have to add the altitude and the radius of the planet to get the value of r.

Clearly from the equation above, the orbital velocity of a satellite depends on its distance from the centre of the planet and the mass of the planet it is orbiting.

For a given planet, the nearer to the planet, the faster the required orbital velocity. For Earth, a satellite at an altitude of 200 kilometres, requires an orbital velocity of a little more than 27,400 km/h. To maintain an orbit that is 35,786 kilometres above Earth’s equator, a satellite must orbit at a speed of about 11,300 km/h. That orbital speed and distance permit the satellite to make one revolution in 24 hours. Since Earth also rotates once in 24 hours, a satellite at this altitude, travelling in the same direction as the Earth’s rotation, stays in a fixed position relative to a point on Earth's surface. Because the satellite stays right over the same spot all the time, this kind of orbit is called **geostationary**. Geostationary orbits are ideal for weather satellites and communications satellites. More on that soon.

In general, the higher the orbit, the longer the satellite can stay in orbit. At lower altitudes, a satellite runs into traces of Earth's atmosphere, which creates drag. The drag causes the orbit to decay (Orbital Decay) until the satellite falls back into the atmosphere and burns up. At higher altitudes, where the vacuum of space is nearly complete, there is almost no drag and a satellite like the Moon can stay in orbit for many millennia.

**Extension – How Long Can A Satellite Stay In Orbit?**

A satellite in any orbit will continue to orbit indefinitely unless acted on by an external force. As mentioned above, satellites in low orbit experience drag from the upper fringes of the atmosphere. This slows the satellite down, resulting in a lowering of the orbital altitude. This can be compensated for by using fuel to maintain the desired altitude, but inevitably, once the fuel runs out, the orbit will degrade, and the satellite will enter the atmosphere and either burn up, or hit the Earth’s surface.

For Geostationary satellites at around 36,000 km altitude, there is no atmosphere to speak of, so there is no drag force and thus the orbital altitude does not decay.

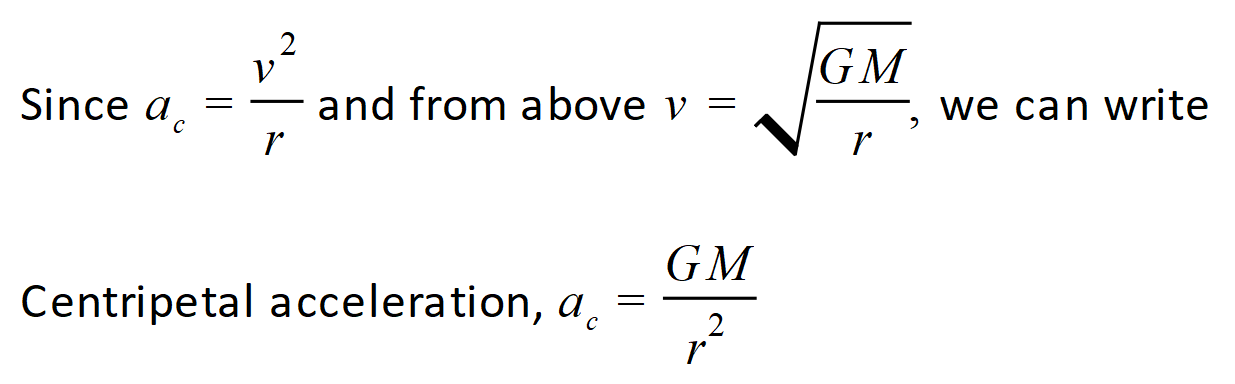
Fuel is still required on such a satellite to maintain its orbital position and shape. All artificial satellites require on-board fuel for this purpose. However, even when this fuel runs out, the satellite will not fall to Earth, but will wander around the orbit, under the pull of external forces such as the Earth's irregular gravitational field, as well as the pull of the Moon and the Sun. This poses a threat to the operational safety of other satellites, and it is for this reason that satellite operators maintain a small reserve of fuel on-board, which is used at the end of life of the satellite to push it up to a higher “graveyard” orbit, so that it will not get in the way of other satellites.

It is interesting to think that, after many millennia have past and the human species and all our Earthly achievements are long gone, our sole remaining monuments may be the remains of those satellites, still endlessly orbiting.

**Centripetal Acceleration of Planets and Satellites In Orbit**

An orbiting satellite of any kind, artificial or natural, moves within a gravity field. Let us consider a satellite in orbit around the Earth. Both the satellite and the Earth attract one another in accordance with Newton’s Law of Universal Gravitation. Earth’s gravitational force is constantly pulling the satellite down toward the centre of the planet. **The centripetal acceleration, ac, experienced by the satellite is produced by the force of gravity.**

If a satellite has insufficient orbital velocity, the gravitational force will win and pull the satellite down onto the earth’s surface.



Clearly, as we would expect, this is the same result we derived for the gravitational field strength, **g**, a distance **r** from the centre of a planet.

**Exercise:** The GOES-17 (Geostationary Operational Environmental Satellite) was launched in 2018 and occupies a geostationary orbit around the Earth.

(a)  Explain the meaning of the term “geostationary orbit”.

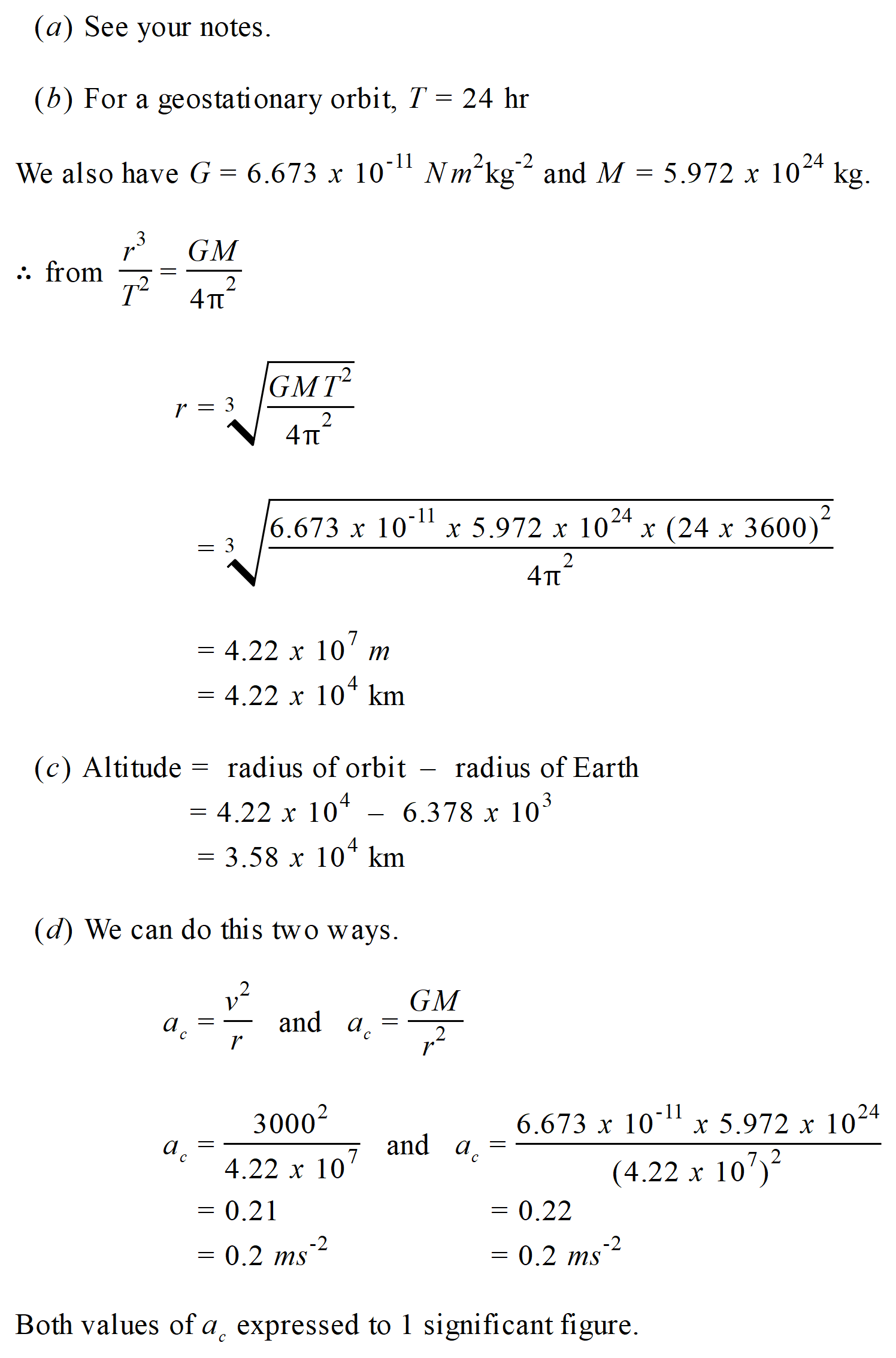
(b) Given that the mass of the earth is 5.972 x 1024 kg and that the universal gravitational constant, G = 6.673 x 10-11 SI units, calculate the radius of the orbit of GOES-17 in kilometres.  (4.22 x 104 km)

(c)  If the earth’s radius is 6378 km at the equator, determine the altitude of GOES-17 in kilometres.  (3.58 x 104 km)

(d) Determine the centripetal acceleration of GOES-17 as it undergoes circular motion around the earth with an orbital velocity of 3000 m/s.  (0.2 ms-2)

Try the above exercise first before looking at the solutions on the next page.

**Solution:**

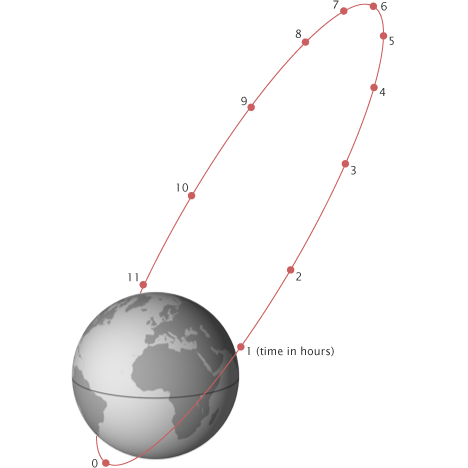


**Types of Orbit**

Satellites are placed in one of several different types of orbit depending on the nature of their mission.  Three common orbit types are **Low Earth Orbit, Medium Earth Orbit** and **Geostationary Orbit**.

Low Earth Orbit (LEO) is defined to be from 160 to 2000 km above the Earth’s surface. Geostationary Orbit (GEO) is defined as one in which the satellite has a circular orbit in the earth’s equatorial plane with a period of 24 hours. A satellite in such an orbit has an altitude of 35786 km and remains above the same point on the equator, making it ideal for observation of a specific location. Medium Earth Orbit (MEO) lies in between LEO & GEO.  The table that follows shows a brief comparison of the features of several types of orbit.

|  |  |  |  |
| --- | --- | --- | --- |
| **Orbit** | **Altitude (km)** | **Characteristics** | **Applications** |
| **LEO** | 160 – 2000 | **Low Earth Orbit**  60 – 90 minute orbital period  Smallest field of view  Speed about 7.8 km/s  Frequent coverage of specific or varied locations – close to ground – good images  Orbital plane can be tilted  Orbits at less than 400 km difficult to maintain due to satellite drag | Military – high resolution photos of Earth’s surface  Earth Observation – satellite imaging  Weather Monitoring  Communications – multiple satellites working together as a network to give constant coverage  International Space Station (ISS)  Hubble Space Telescope |
| **MEO** | 2000 – 35786 | **Medium Earth Orbit**  Sometimes called intermediate circular orbit (ICO)  Any inclination to the equatorial plane is possible for these orbits  Less orbital perturbations & less orbital drag than for LEO, meaning less corrections needed to orbital path  Larger field of view than for LEO and cheaper to achieve than GSO or GEO (see below)  Most commonly used altitude for MEO is 20200 km which gives orbital period of 12 hours – satellite is over same places on Earth twice per day – important to keep ground track (the path over the Earth’s surface traced by satellite) the same each day for GPS satellites, as this allows better management of errors in the satellites’ positions. | Navigation - Global Positioning Satellites  Geodetic/space environment monitoring  Communications covering north & south poles |
| **GSO** | 35 786 km | **Geosynchronous Earth Orbit**  Matches the Earth’s rotation - Orbital period is 24 hours  Orbits from west to east following Earth’s rotation at any inclination to the equatorial plane  Speed about 3 km/s  From the ground, can appear to move in the sky but returns to the same position in the sky after 24 hours.  Maintains consistent position over a single longitude. | Telecommunications  Mass-media  Weather Monitoring – Meteorology  Remote Sensing |
| **GEO** | 35 786 | **Geostationary Earth Orbit**  Type of GSO but satellite orbits in the **equatorial plane** from west to east following Earth’s rotation  Remains fixed above one point on Earth – allows fixed antennas on Earth to stay pointed at the satellite  Speed about 3 km/s  Allows tracking of stationary point on earth  Largest field of view – as few as 3 equally spaced satellites can provide near global coverage  Check out:  <https://gisgeography.com/geosynchronous-geostationary-orbits/> | Telecommunications  Mass-media  Weather Monitoring – Meteorology  Remote Sensing |
| **HEO** | Varies | **Highly Elliptical Orbit**  Elliptic orbit with high eccentricity (very non-circular).  Orbital velocity varies.  Satellite moves slowly through a long apogee and remains at high altitude over high-latitude ground sites for long periods of time. This makes these elliptical orbits useful for communications satellites.  Examples – Molniya & Tundra orbits  See diagram below. | Communications, especially at high latitudes.  Remote sensing |



A Molniya Orbit – in this example of a Highly Elliptical Orbit, we see that there is a long period of time for satellite transmission to the northern hemisphere, especially to the high latitudes. (Diagram taken from: <https://en.wikipedia.org/wiki/Molniya_orbit> .)

There are many other examples of different types of orbits for satellites. Each is used to achieve specific desired outcomes for the satellite mission. Check out the following websites for further explanation & diagrams of different types of orbits.

[Types of Orbits I](https://earthobservatory.nasa.gov/features/OrbitsCatalog)

[Types of Orbits II](https://www.esa.int/Enabling_Support/Space_Transportation/Types_of_orbits#GEO)

[Types of Orbits III](https://www.spacefoundation.org/space_brief/types-of-orbits/)

**Elliptical Orbits**

It is important to clarify that we can only use circular motion mechanics to describe and explain satellite motion that occurs in circular orbits. Most satellites are placed in orbits that are very close to circular, so we can use our circular motion equations to gain a reasonably accurate understanding of their motions.

Elliptical orbits, however, cannot be described accurately using circular motion equations. The circular motion equations need to be modified slightly to deal with elliptical orbits. Such modifications are beyond the scope of the current syllabus. Elliptical motion equations are not listed for study in this syllabus.

It is still required that you understand the differences between circular and elliptical orbits on a qualitative basis. For circular orbits, the orbital velocity is constant and therefore so too is the kinetic energy of the satellite. For elliptical orbits, this is not the case.

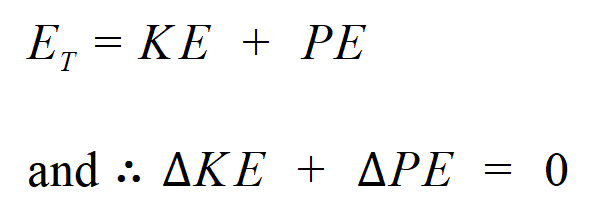
As mentioned above in our example of the Molniya Orbit, a satellite in an elliptical orbit has a **non-constant orbital velocity** and therefore a **non-constant kinetic energy**. The satellite has maximum velocity at perigee (closest point to Earth) and minimum velocity at apogee (furthest point from Earth). That is, the satellite moves more quickly when closest to the Earth and more slowly when furthest from the Earth. This is predicted by Kepler’s Law of Areas.

The other qualitative fact to be aware of is the concept of eccentricity. The **eccentricity** of an astronomical object or orbit is a dimensionless parameter that determines the amount by which the orbit deviates from a perfect circle. This has been previously explained in the section of notes on Kepler’s 1st Law. There is a defining equation for eccentricity in terms of the variables that define an ellipse. This equation is not required by the current syllabus.

**Gravitational Potential Energy (GPE)**

Gravity fields are examples of **conservative force fields**. A conservative force is one for which the work done by the force on a particle depends only on the position of the particle in the field and not on the path taken by the particle through the field.

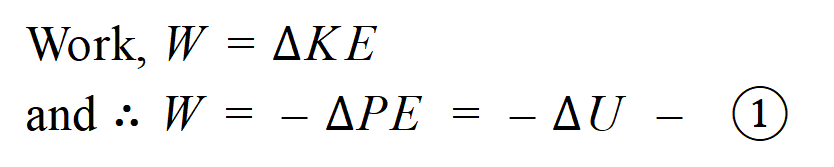
For a conservative field (or system), the total energy, **ET**, remains constant.



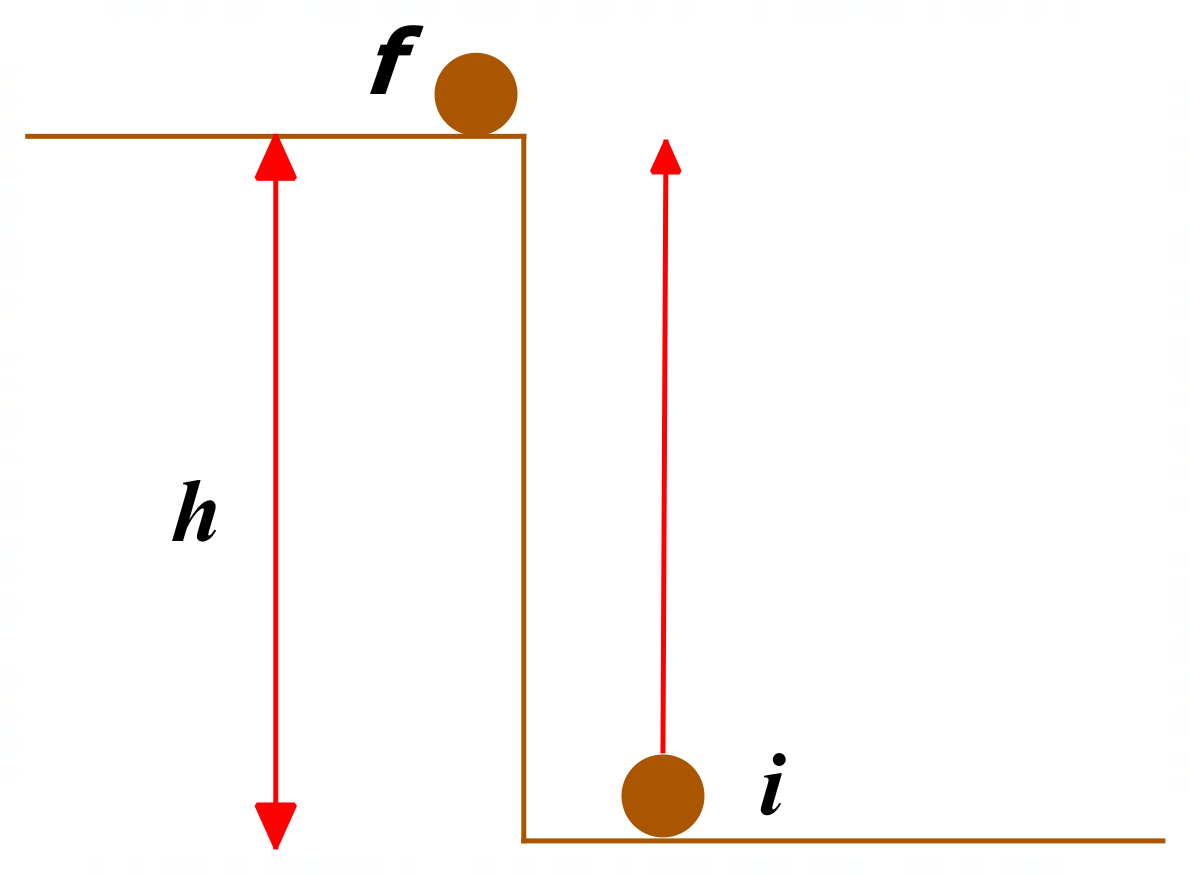
KE = kinetic energy of particle

PE = potential energy of particle, which in this case is **gravitational potential energy**, usually denoted by the symbol **U**.

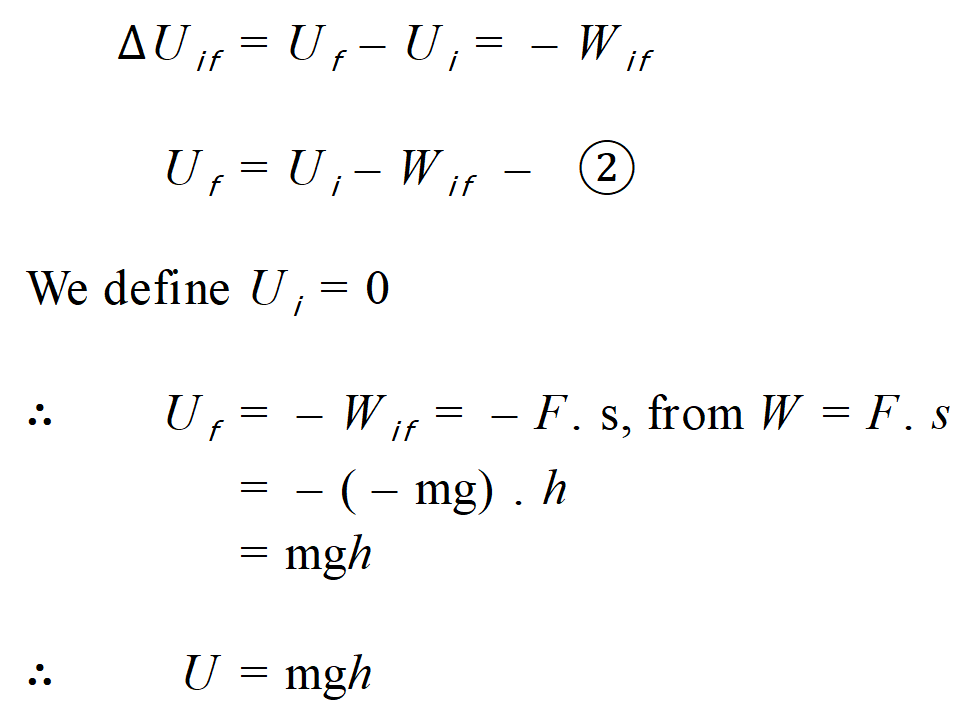
From the Dynamics Module in Year 11, we know that the work done by the resultant force acting on a particle is equal to the change in kinetic energy of the particle.



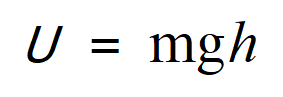
Let us first consider the case when the particle remains close to the surface of the Earth. Consider a particle that is lifted from an initial position, **i**, on the ground to a final position, **f**, a distance **h** above the ground.



When we lift an object from the ground to a particular height above the ground we must do work against the gravitational field of the Earth.  This work goes into increasing the gravitational potential energy of the body.  The amount of work done is equal to the change in gravitational potential energy (GPE) of the body, as shown by equation 1 above.



We find that when an object of mass, **m**, is close to the Earth’s surface, where the acceleration due to gravity, **g**, is fairly constant, the gravitational potential energy of the object at height, **h**, above the ground is given by:

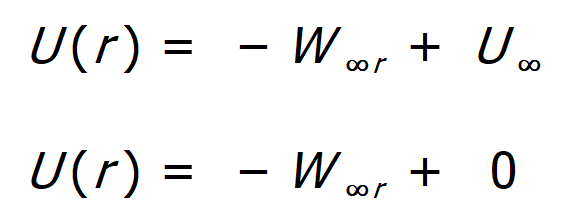


Note that we have defined the ground level as the zero of potential energy.

Next, we need to ask: what happens in the case where the object in question moves over large distances from the surface of Earth, **where g is not constant**? What is the equation for the gravitational potential energy of the object in this case?

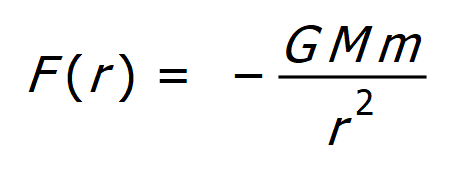
To derive the equation for this case, we need to use a small amount of calculus. There really is no other way. We also need to redefine the zero of potential energy to occur where the object and the Earth are infinitely separated. This is necessary because of the inverse square relationship between gravitational force and distance of separation. The gravitational force between the Earth and another object only drops to zero at an infinite distance. So, if we need to find a place where the object has no potential energy, that can only be at **r = ∞**.

So, from equation 2 above, when an object of mass, **m**, is distant, **r**, from the centre of the Earth, the gravitational potential energy of the object is given by:

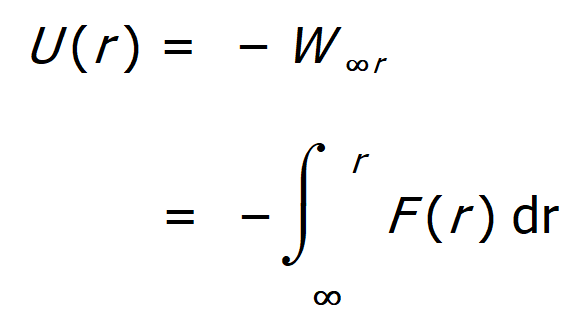


Where **W∞ r** = work done by gravity on the object as it moves from infinity to a distance **r** from the centre of the Earth.

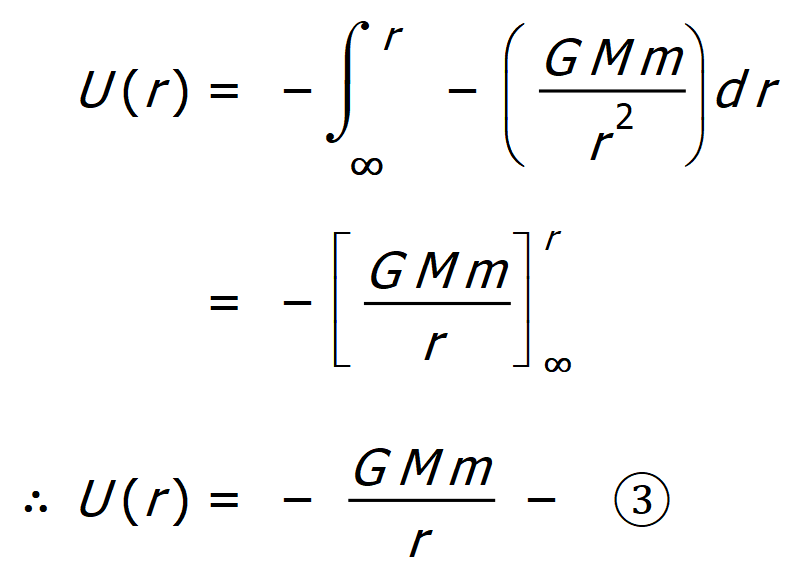
Assume for simplicity that the object moves toward Earth of mass, **M**, along a radial line. The gravitational force **F(r)** acting on the object is:



where, the minus sign indicates an attractive force that pulls the object towards Earth. Then:



That is, we “add up” all the work, **F(r) dr**, done by gravity in moving the object from **∞** to **r**. The integral is needed because the force is not constant. The force on the object increases as the object moves from **∞** to **r**.



Equation 3 above gives the **total potential energy**, **U**, of a planet or satellite of mass, **m**, in orbit around a central body of mass, **M**. **r** is the centre to centre distance between the central body and the planet or satellite in orbit.

**Note that the r in the denominator is not squared.**

The minus sign indicates that the potential energy is negative at any finite distance. That is, **U = 0** at **r = ∞** and decreases as the separation distance decreases. This corresponds to the fact that the gravitational force exerted by the Earth on the object is attractive. As the object moves from **r = ∞** toward the Earth, the work **W∞ r** done by this force on the object is positive, which means from equation 2 above, that **U(r)** is negative.

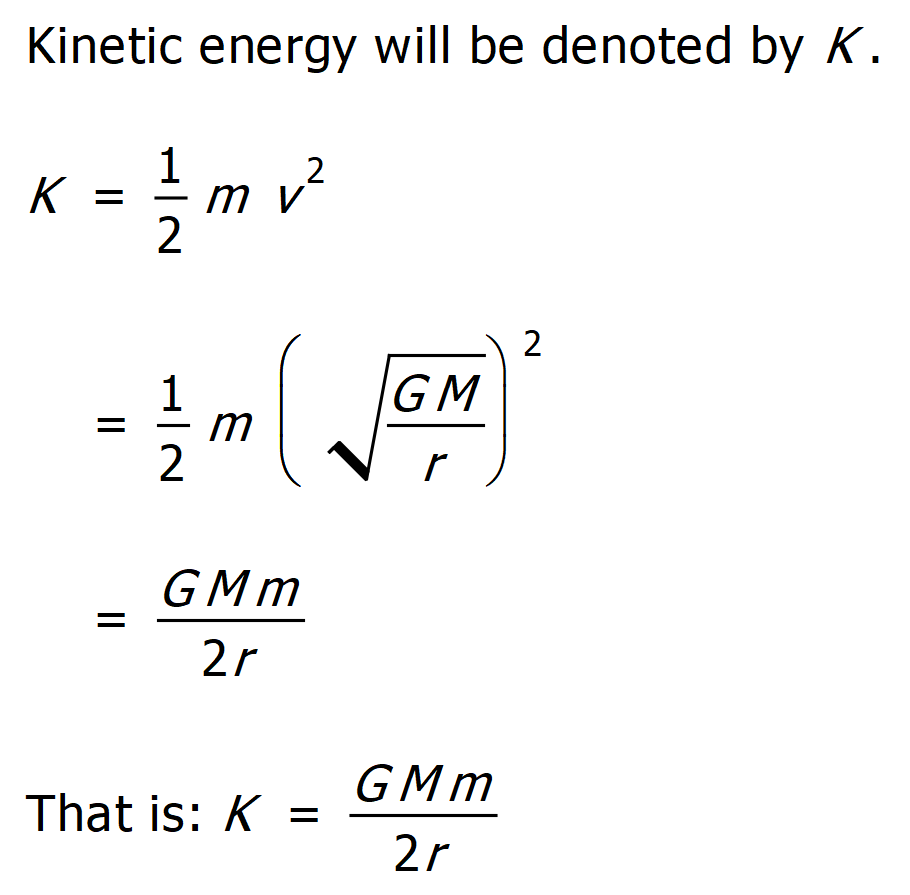
Equation 3 holds regardless of the path taken by the object in moving from **∞** to **r**, because gravity is a conservative force.

This formulation is very convenient for describing the energy requirements for travelling between different bodies in the solar system. We can imagine coming in for landing on a planet. As we come closer to the planet, we gain kinetic energy. Since energy is conserved, we lose gravitational potential energy to account for this. Thus, **U**, becomes more negative.

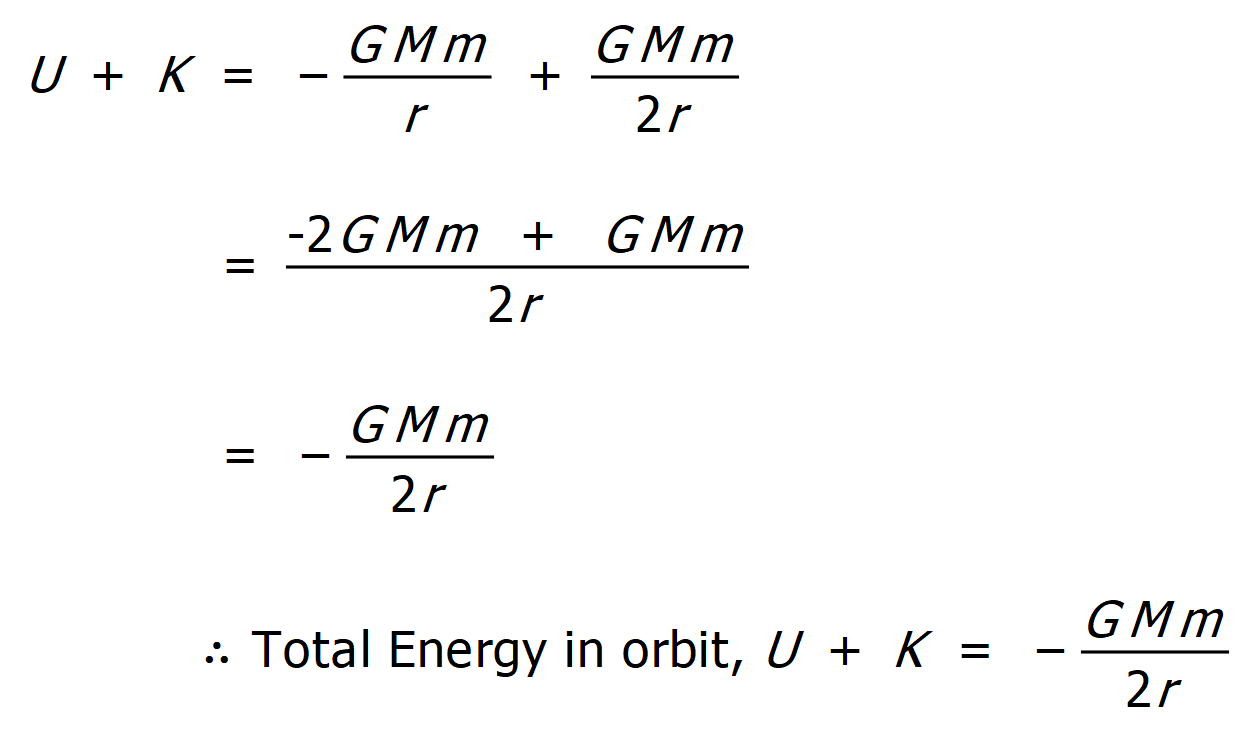
**Exercise:** Determine the GPE for a satellite of mass 200 kg launched from the surface of Mars into an orbit 650 km above the planet’s surface.  (**Data:** radius of Mars = 3.4 x 106 m, mass of Mars = 6.5 x 1023 kg, G = 6.67 x 10-11 SI Units.  **Answer:** EP =  - 2.14 x 109 J)

**Total Energy In Orbit**

We have an equation already for the orbital velocity of an object, so we can quickly derive an expression for the **total kinetic energy** of the object in orbit.

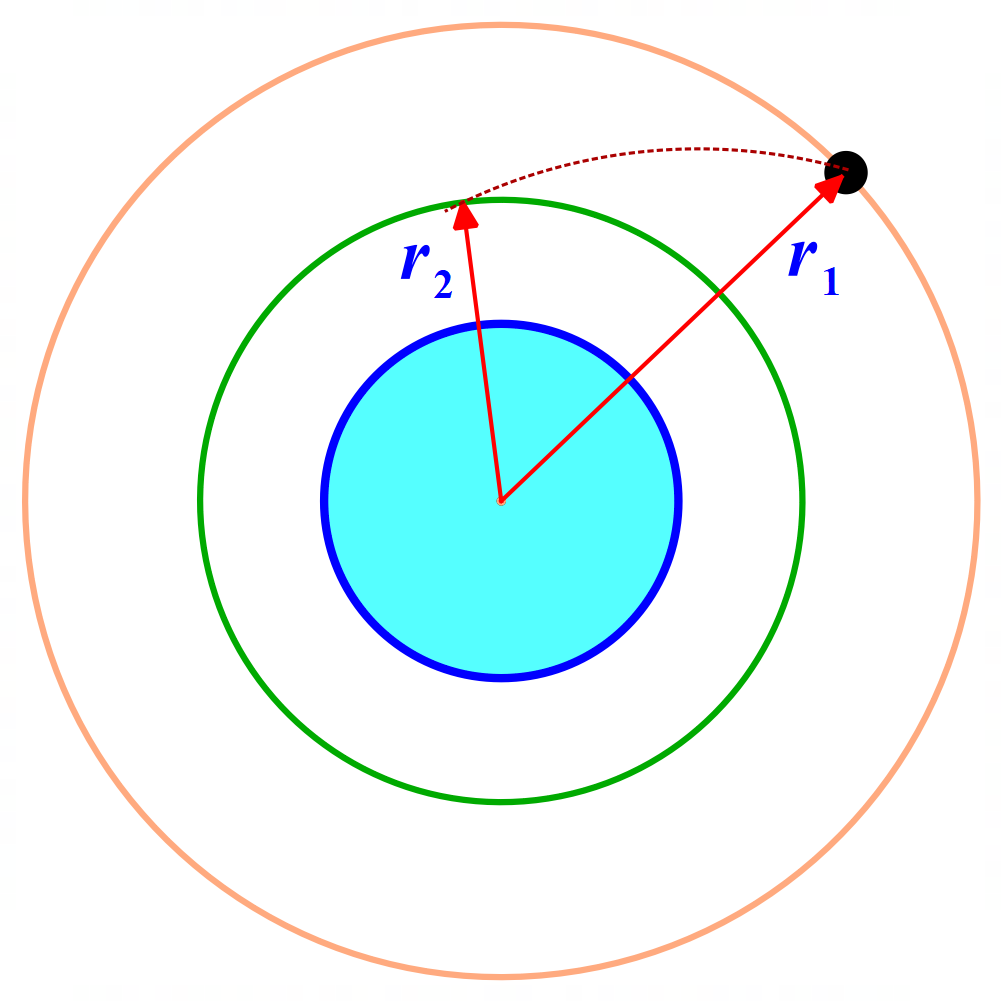
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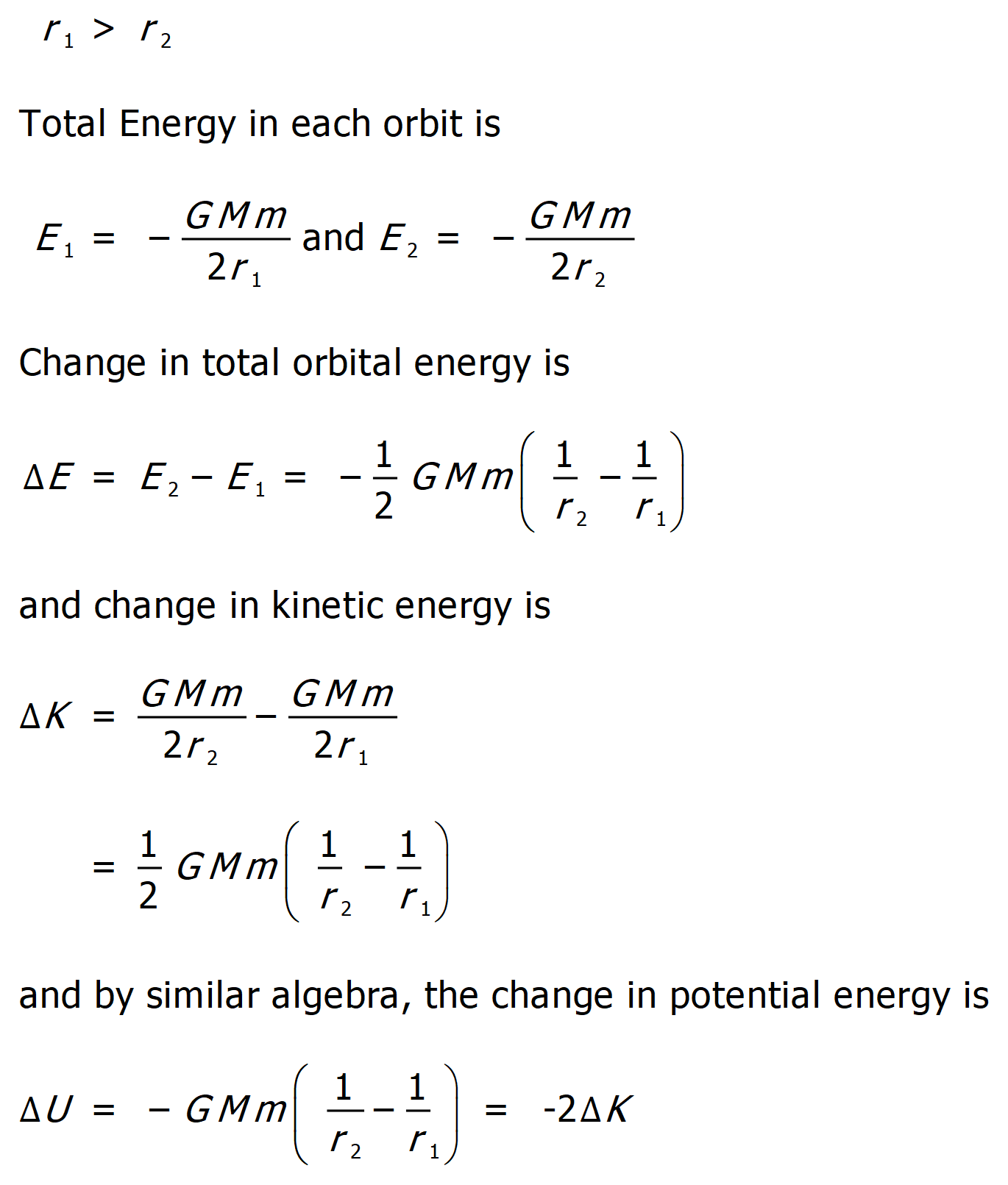
We are now able to derive the expression for the **total energy of an object in orbit**.



**Energy Changes Between Orbits**

When an object is in a stable circular orbit the total energy of the object is constant. What happens to the energy of an object when it moves between orbits? Consider a satellite moving from one orbit to another as shown below.





So, what do these equations tell us about the energy changes that occur when satellites move between orbits? Since **r2** is smaller than **r1**, the change in kinetic energy of the satellite is positive. The kinetic energy increases as the satellite enters the lower orbit. Its orbital velocity also increases.

The change in potential energy of the satellite is negative. Potential energy decreases, as expected, and decreases twice as much as the kinetic energy increases. So, the total energy of the satellite also decreases, as indicated by the change in total energy equation above.

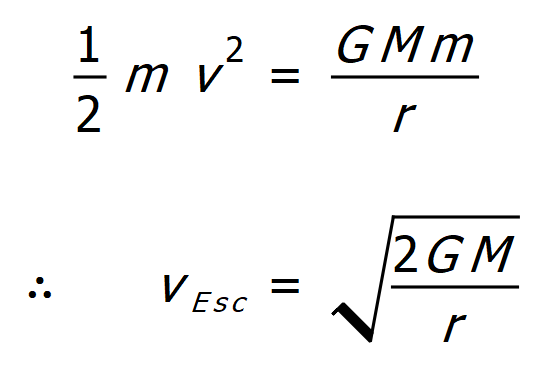
**Extension – How Does A Satellite Move To A Lower Orbit?**

The fact that the change in total orbital energy of a satellite is negative when it moves from a higher to a lower orbit, tells us that the satellite must do work to change its motion. It does this by firing rockets in a direction opposite to its orbital motion. This slows the satellite down so that it can no longer maintain its current orbit. Gravity then pulls the satellite radially inwards increasing its kinetic energy. With further help from its rockets to correct its path, the satellite is guided into the required lower orbit and is then travelling at the correct higher orbital velocity to maintain that orbit.

**Escape Velocity**

How fast must an object be launched from a planet’s surface so that it escapes from the planet’s gravitational attraction?

While ever an object is in the planet’s gravity field, it has a potential energy due to its position in the field. So, if we were to remove the object to the place where it no longer has any potential energy due to the field, the object would be free from the field. Thus, we must supply enough kinetic energy to the object to change its potential energy from its value at the planet’s surface to zero.



This is the equation for **escape velocity**, defined as the initial velocity of an object that can escape the surface of a moon or planet of mass, **M**, and radius **r**.

**Exercise:** Determine the escape velocity for Earth, given mass of Earth = 5.972 x 1024 kg, radius of Earth = 6.378 x 103 km and G = 6.673 x 10-11 SI units. (**Answer:** 1.118 x 104 m/s or 11.2 km/s)

**Extension**

As stated above, escape velocity can be defined as the initial velocity of an object that can escape the surface of a moon or planet. More generally, it is the speed at ***any* position** in the gravitational field, such that the ***total* energy** at that point becomes zero. If the total energy is zero or greater, the object escapes the gravitational force holding it in orbit. If the total energy is negative, the object cannot escape. Let’s see why that is the case.

As noted earlier, we see that **U→0** as **r→∞**. If the total energy is zero, then as the object ***m*** reaches a value of **r** that approaches infinity, the potential energy, **U,**becomes zero and so must the kinetic energy, **K**. Hence, ***m*** comes to rest infinitely far away from ***M***. It has “just managed to escape” ***M***. If the total energy is positive, then the object still has **K** at **r = ∞** and certainly ***m*** does not return. When the total energy is zero or greater, then we say that ***m*** is **not gravitationally bound** to ***M***.

On the other hand, if the total energy is negative, then the kinetic energy must reach zero at some finite value of *r*, where the potential energy is negative and equal to the total energy. The object can never exceed this finite distance from ***M***, since to do so would require the kinetic energy to become negative, which is not possible. We say ***m*** is **gravitationally bound** to ***M***.

**Concluding Comment**

We have now covered all the theory aspects of this module. Obviously, your teacher will present you with opportunities for practical work & further question & problem work. The understanding of Mechanics is extremely useful Physics that has innumerable applications in the real world. There is much more Mechanics for you to learn in Physics courses at university level. I hope you have enjoyed the taste of this exciting area of Physics that has been presented in the current Syllabus.

**APPENDIX A**

**Statement of Syllabus Content Covered in these Notes**

The following indicates the specific content from the **Stage 6 Physics Syllabus** that has been covered in the notes, worksheets & practicals provided on the Advanced Mechanics Module web page.

The resources on this website are meant to supplement the work you do in class NOT replace it. The notes will always provide you with a comprehensive and accurate set of notes on the Module under study. The worksheets will provide some introduction & practice to appropriate problem-solving skills for the topic. You will need to do much more problem-solving practice than just what is provided on this website. The practicals section will provide some experiments relevant to the topic but again you will need to do more than just what is suggested here. Your teacher should provide you with much more problem-solving & practical experience than you will find on this website.

The content statements that are **ticked** have been covered. Those left without a tick have either not been covered at all or have been only partially covered. These are mainly content statements requiring practical work of some kind.

### Content

#### Projectile Motion

**Inquiry question:** How can models that are used to explain projectile motion be used to analyse and make predictions?

Students:

* analyse the motion of projectiles by resolving the motion into horizontal and vertical components, making the following assumptions: ✓
  + a constant vertical acceleration due to gravity ✓
  + zero air resistance ✓
* apply the modelling of projectile motion to quantitatively derive the relationships between the following variables: ✓
  + initial velocity ✓
  + launch angle ✓
  + maximum height ✓
  + time of flight ✓
  + final velocity ✓
  + launch height ✓
  + horizontal range of the projectile (ACSPH099) ✓
* conduct a practical investigation to collect primary data in order to validate the relationships derived above.
* solve problems, create models and make quantitative predictions by applying the equations of motion relationships for uniformly accelerated and constant rectilinear motion  Information and communication technology capability icon Numeracy icon ✓

#### Circular Motion

**Inquiry question**:Why do objects move in circles?

Students:

* conduct investigations to explain and evaluate, for objects executing uniform circular motion, the relationships that exist between: ✓
  + centripetal force ✓
  + mass ✓
  + speed ✓
  + radius ✓
* analyse the forces acting on an object executing uniform circular motion in a variety of situations, for example: ✓
  + cars moving around horizontal circular bends ✓
  + a mass on a string ✓
  + objects on banked tracks (ACSPH100) Critical and creative thinking icon  Information and communication technology capability icon ✓
* solve problems, model and make quantitative predictions about objects executing uniform circular motion in a variety of situations, using the following relationships: ✓
  + ✓
  +  Information and communication technology capability icon Numeracy icon ✓
  + ✓
* investigate the relationship between the total energy and work done on an object executing uniform circular motion ✓
* investigate the relationship between the rotation of mechanical systems and the applied torque ()  Information and communication technology capability icon Numeracy icon ✓

#### Motion in Gravitational Fields

**Inquiry question**: How does the force of gravity determine the motion of planets and satellites?

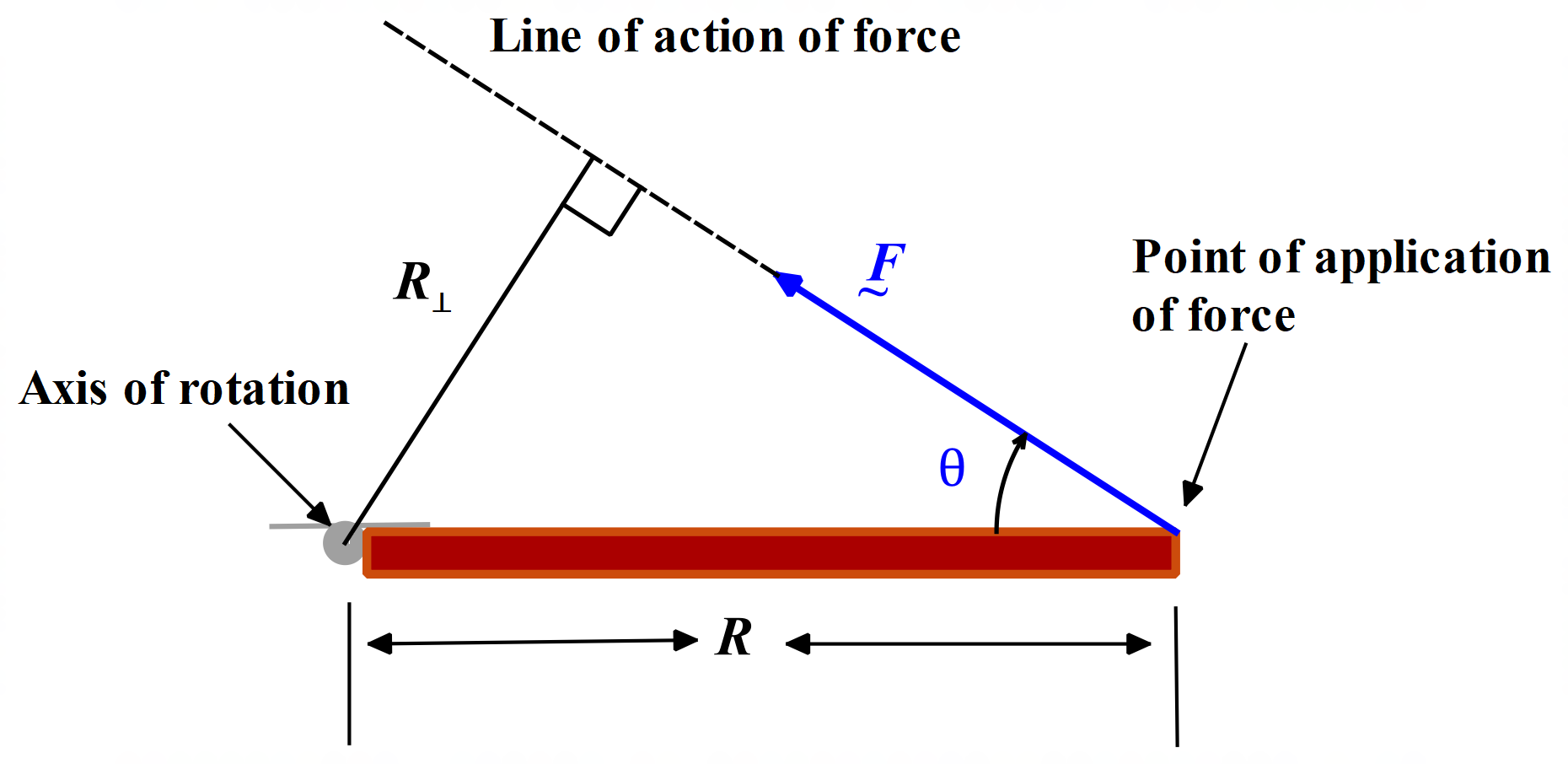
Students:

* applyqualitatively and quantitatively Newton’s Law of Universal Gravitation to:
  + determine the force of gravity between two objects) ✓
  + investigate the factors that affect the gravitational field strength () ✓
  + predict the gravitational field strength at any point in a gravitational field, including at the surface of a planet (ACSPH094, ACSPH095, ACSPH097) ✓
* investigate the orbital motion of planets and artificial satellites when applying the relationships between the following quantities: Critical and creative thinking icon  Information and communication technology capability icon Numeracy icon ✓
  + gravitational force ✓
  + centripetal force ✓
  + centripetal acceleration ✓
  + mass ✓
  + orbital radius ✓
  + orbital velocity ✓
  + orbital period ✓
* predict quantitatively the orbital properties of planets and satellites in a variety of situations, including near the Earth and geostationary orbits, andrelate these to their uses (ACSPH101)  Information and communication technology capability icon Numeracy icon ✓
* investigate the relationship of Kepler’s Laws of Planetary Motion to the forces acting on, and the total energy of, planets in circular and non-circular orbits using: (ACSPH101) ✓
  + ✓
  +  Information and communication technology capability icon Numeracy icon ✓
* derive quantitatively and apply the concepts of gravitational force and gravitational potential energy in radial gravitational fields to a variety of situations, including but not limited to:  Information and communication technology capability icon Numeracy icon ✓
  + the concept of escape velocity () ✓
  + total potential energy of a planet or satellite in its orbit () ✓
  + total energy of a planet or satellite in its orbit () ✓
  + energy changes that occur when satellites move between orbits (ACSPH096) ✓
  + Kepler’s Laws of Planetary Motion (ACSPH101) ✓

**APPENDIX B**

**Alternative Derivation of Torque Formula**

Examine the diagram below.



In the diagram we have drawn in the perpendicular distance from the axis of rotation to the line of action of the force applied to the edge of the door. Clearly, this perpendicular distance is: **R⊥ = R sinθ**.

When we write the expression for torque this time, it becomes:



This is what we found previously. ** = r F sinθ**